Collection of Real Analysis Problems

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Thanks to every Cat I have met in my life

A Note to the Reader

This collection consists of **666 problems in real analysis**, carefully collected over the years from a wide range of mathematical sources. The problems have been gathered from international and national mathematical undergraduate competitions, mathematical journals and magazines, university entrance exams, problem books, and various other training materials.

The problems are collected from sources including the **International Mathematics Competition (IMC)**, **William Lowell Putnam Mathematical Competition**, **American Mathematical Monthly**, **Crux Mathematicorum**, **Romanian Mathematical Magazine**, **Gazeta Matematica**, **The College Mathematics Journal**, **Mathematical Reflections Journal**, **Romanian District Olympiad**, **Romanian National Mathematical Olympiad**, **Jozsef Wildt International Mathematical Competition**, **Vojtech Jarnik International Mathematical Competition**, **OLYMPIC Toán Sinh viên toàn quoc**, **Berkeley Mathematics Competition**, **ISI/CMI Entrance Exams**, **Iranian Mathematical Competition** as well as various other undergraduate mathematics competitions and problem collections.

I am deeply grateful to all the authors, problem setters, competition organisers, editors, members of AOPS, and other mathematical communities whose work made this collection possible. Their contributions have provided an invaluable source of challenging and interesting problems for students and enthusiasts of mathematics.

Since several problems have been translated from Romanian and Vietnamese with the assistance of Claude AI, there may be inaccuracies or ambiguities arising from translation or interpretation. Every effort has been made to preserve the mathematical meaning of the original problems; however, errors in translation, notation, or editing may still remain.

If you find any errors, inconsistencies, or omissions, or if you have a correction or a suggestion for a new problem, please feel free to contact me by email at:

`category.theory.for.cats@gmail.com`

If you are interested in contributing to this collection by providing solutions to the problems, you are welcome to contribute through the GitHub repository:

Collection of Real Analysis Problems

The latest PDF version of this handout is available at:
`https://zetazero-0.github.io/notes.html`

1. Let $f_n : \mathbb{R} \to \mathbb{R}$ $(n \geq 1)$ be the sequence of functions with general term

$$f_n(x) = \frac{n^2 x}{x^2 + n^2}.$$

a) Prove that the sequence (f_n) converges pointwise on $\mathbb{R}$ and determine its pointwise limit f.
b) Show that (f_n) does not converge uniformly on $\mathbb{R}$.
c) Show that $\displaystyle\int_0^x f(t)\,dt = \lim_{n\to\infty} \int_0^x f_n(t)\,dt$ for every $x \in \mathbb{R}$.

2. Given a nonempty set $A \subseteq \mathbb{R}$, a function $f : A \to \mathbb{R}$ is called a *Baire function* if there exists a sequence of functions $f_n : A \to \mathbb{R}$ $(n \geq 1)$, continuous on A, which converges pointwise to f on A.
a) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$, then its derivative f' is a Baire function.
b) Give an example of a Baire function that is not the derivative of any function.

3. Let $f_n : [a, b] \to \mathbb{R}$ $(n \geq 1)$ be a sequence of functions satisfying:
(i) all functions f_n are differentiable on $[a, b]$;
(ii) the sequence (f_n) converges pointwise on $[a, b]$ to a differentiable function $f : [a, b] \to \mathbb{R}$;
(iii) the sequence (f_n') converges pointwise on $[a, b]$ to a continuous function $g : [a, b] \to \mathbb{R}$.
Prove that $f'(x) = g(x)$ for all $x \in [a, b]$.

4. Let $g : [0, 1] \to \mathbb{R}$ be a continuous function with $g(1) = 0$, and let $f_n : [0, 1] \to \mathbb{R}$ $(n \geq 1)$ be the sequence of functions defined by $f_n(x) := x^n g(x)$ for every $n \geq 1$ and every $x \in [0, 1]$. Prove that the sequence (f_n) converges uniformly on $[0, 1]$.

5. Let $A \subseteq \mathbb{R}$ be a nonempty set, and let $f_n : A \to \mathbb{R}$ $(n \geq 1)$ be a sequence of uniformly continuous functions which converges uniformly to $f : A \to \mathbb{R}$. Prove that f is uniformly continuous.

6. Let $f_n : \mathbb{R} \to \mathbb{R}$ $(n \geq 1)$ be a sequence of functions that converges uniformly on $\mathbb{R}$ to the function $f : \mathbb{R} \to \mathbb{R}$. Suppose that for each $n \geq 1$ there exists $\ell_n = \lim_{x\to\infty} f_n(x) \in \mathbb{R}$. Prove that the limits $\lim_{n\to\infty} \ell_n$ and $\lim_{x\to\infty} f(x)$ both exist and are equal.

7. Let $f_1 : [a,b] \to \mathbb{R}$ be a Riemann integrable function, and let $f_n : [a,b] \to \mathbb{R}$ $(n \geq 1)$ be the sequence of functions defined recursively by

$$f_{n+1}(x) = \int_a^x f_n(t)\,dt \quad \text{for every } x \in [a,b] \text{ and every } n \geq 1.$$

Prove that the sequence (f_n) converges uniformly to the zero function on $[a,b]$.

8. Let $f_n : [0,1] \to \mathbb{R}$ $(n \geq 1)$ be the sequence defined recursively by $f_1(x) = 0$,

$$f_{n+1}(x) = f_n(x) + \frac{1}{2}\left(x - f_n^2(x)\right) \quad \text{for every } x \in [0,1].$$

Prove that the sequence (f_n) converges uniformly on $[0,1]$ to the function $f(x) = \sqrt{x}$.

9. Evaluate each of the following limits/integrals.

(a) Calculate $\displaystyle\lim_{n\to\infty} \int_0^1 e^{x^n}\,dx$.

(b) Determine $\displaystyle\lim_{n\to\infty} n \int_{-1}^0 (x + e^x)^n\,dx$.

(c) Determine $\displaystyle\lim_{n\to\infty} n \int_0^1 (\cos x - \sin x)^n\,dx$.

(d) Determine

$$\lim_{n\to\infty} \left(\int_0^1 e^{x^2/n}\,dx\right)^n.$$

(e) Show that

$$\lim_{n\to\infty} \int_0^3 \frac{x^2(1-x)x^n}{1+x^{2n}}\,dx = 0.$$

(f) Let $n \in \mathbb{N}$. Evaluate the integral

$$\int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin(x)}\,dx.$$

(g) Prove that the limit

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{\cos^n(\pi x)}{(x-n)^2+1}\,dx$$

exists and determine its value.

(h) Evaluate

$$\lim_{n\to\infty} \left(1 + \int_0^1 \frac{x^n \ln(1+x)}{1+x^n}\,dx\right)^n.$$

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Prove that there exists a sequence of polynomial functions $f_n : \mathbb{R} \to \mathbb{R}$ $(n \geq 1)$, uniformly convergent to f, if and only if f is a polynomial function.

11. Let $m \in \mathbb{N}$ and let $f_n : [0,1] \to \mathbb{R}$ $(n \geq 1)$ be a sequence of polynomial functions of degree at most m, which converge pointwise to the function $f : [0,1] \to \mathbb{R}$. Prove that f is a polynomial function of degree at most m, and that the sequence (f_n) converges uniformly to f on $[0,1]$.

12. Determine the convergence of the series

$$\sum_{n\geq 1} \left(\frac{2n^2+5}{7n^2+3n+2}\right)\left(\frac{x}{2x+1}\right)^n, \qquad x \in \mathbb{R} \setminus \left\{-\tfrac{1}{2}\right\}.$$

13. Define the sequence $x_1 = a \in \mathbb{R}$; $x_{n+1} = x_n^2 + x_n + 1$. Calculate

$$\lim_{n\to\infty} \frac{(x_{n+1}+1)^2}{(x_1^2+2)(x_2^2+2)\cdots(x_n^2+2)}.$$

14. Consider the sequence $(x_n)_{n\geq 0}$ defined by $x_0 > 0$; $x_{n+1} = \dfrac{x_n}{1+nx_n^2}$. Study the convergence of the sequences $(x_n)_{n\geq 0}$ and $(nx_n)_{n\geq 0}$, and determine their possible limits.

15. Consider the sequence (x_n) of real numbers, defined by $x_1 > 0$; $x_1 + x_1^2 < 1$ and $x_{n+1} = x_n + \dfrac{x_n^2}{n^2}$ for every $n \geq 1$. Show that the sequences $\left(\dfrac{1}{x_n} - \dfrac{1}{n-1}\right)$ and (x_n) are convergent.

16. Let $(a_n)_{n\geq 1}$ be a sequence with strictly positive terms. If

$$a_n \leq a_{2n} + a_{2n+1}, \qquad \forall n \in \mathbb{N}^*,$$

study the convergence of the series $\displaystyle\sum_{n=1}^{\infty} a_n$.

17. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x) + f''(x) = -x\, g(x)\, f'(x),$$

where $g : \mathbb{R} \to \mathbb{R}$ satisfies $g(x) > 0$ for all $x \in \mathbb{R}$. Show that f is bounded.

18. Calculate

$$\lim_{n\to\infty}\int_0^1 \sqrt{\frac{x}{2}+\sqrt{\frac{x}{2}+\cdots+\sqrt{\frac{x}{2}+\sqrt{2}}}}\,dx$$

with n radicals in the nested expression.

19. Consider a sequence $(u_n)_{n\geq 1}$ defined by initial terms $u_1, u_2 \in (0,1)$ and the recursive relation

$$u_{n+2} = \frac{1}{n}u_{n+1}^2 + \frac{n-1}{n}\sqrt{u_n} \quad \text{for all } n \in \mathbb{N}.$$

Determine the limit of the sequence (u_n) as $n \to \infty$.

20. Let the sequence (u_n) be defined by $u_1 = \dfrac{1}{2}$ and

$$u_{n+1} = u_n + \frac{u_n^2}{n^2}, \qquad \forall n \in \mathbb{N}^*.$$

a) Prove that $u_n + n^2 \geq \dfrac{n(n+1)}{\sqrt{2}}$.
b) Prove that (u_n) has a finite limit.

21. a) Calculate $\displaystyle\int_0^1 x\sin(\pi x^2)\,dx$.
b) Calculate $\displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1} k\int_{k/n}^{(k+1)/n} \sin(\pi x^2)\,dx$.

22. Let $f:[0,1]\to(0,\infty)$ be a continuous function.
a) Prove that for every integer $n \geq 1$ there exists a unique partition $(a_0, a_1, \ldots, a_n)$ of $[0,1]$ such that

$$\int_{a_k}^{a_{k+1}} f(t)\,dt = \frac{1}{n}\int_0^1 f(t)\,dt \quad \text{for all } k \in \{0,1,\ldots,n-1\}.$$

b) For each integer $n \geq 1$ define $\bar{a}_n = \dfrac{a_1+\cdots+a_n}{n}$, where $(a_0, a_1, \ldots, a_n)$ is the unique partition from part a). Show that the sequence $(\bar{a}_n)_{n\geq 1}$ is convergent and determine its limit.

23. Let $f : [0,1] \to (0,\infty)$ be a continuous function. For each $n \in \mathbb{N}$, $n \geq 2$, consider the partition $(t_0, t_1, \ldots, t_n)$ of $[0,1]$ with the property that

$$\int_{t_0}^{t_1} f(t)\,dt = \int_{t_1}^{t_2} f(t)\,dt = \cdots = \int_{t_{n-1}}^{t_n} f(t)\,dt.$$

Determine $\displaystyle\lim_{n\to\infty} n\left(\frac{1}{f(t_1)} + \frac{1}{f(t_2)} + \cdots + \frac{1}{f(t_n)}\right)$.

24. Let $f : [a,b] \to \mathbb{R}$ be a differentiable function with Riemann integrable derivative. For each $n \in \mathbb{N}$ denote

$$A_n := \frac{b-a}{n}\sum_{k=0}^{n-1} f\left(a + k\,\frac{b-a}{n}\right), \qquad B_n := \frac{b-a}{n}\sum_{k=1}^{n} f\left(a + k\,\frac{b-a}{n}\right).$$

Prove that

$$\lim_{n\to\infty} n\left(A_n - \int_a^b f(x)\,dx\right) = \frac{b-a}{2}\big(f(a) - f(b)\big)$$

and

$$\lim_{n\to\infty} n\left(B_n - \int_a^b f(x)\,dx\right) = \frac{b-a}{2}\big(f(b) - f(a)\big).$$

25. Determine $\displaystyle\lim_{n\to\infty} n\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} - \ln 2\right)$.

26. Let $f : [0,1] \to \mathbb{R}$ be an increasing, twice differentiable function, with continuous second derivative on $[0,1]$, and let (A_n), (B_n) be the sequences defined by

$$A_n := \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right), \qquad B_n := \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right).$$

The interval $[A_n, B_n]$ is divided into three equal segments. Prove that for n sufficiently large, the number $I := \displaystyle\int_a^b f(x)\,dx$ belongs to the middle segment.

27. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with continuous derivative, and let

$$s_n = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \quad \text{for every } n \in \mathbb{N}.$$

Prove that the sequence $(s_{n+1} - s_n)_{n\geq 1}$ converges to $\displaystyle\int_0^1 f(x)\,dx$.

28. Let $f, g : [a, b] \to \mathbb{R}$ be functions with the property that there exist $\alpha \geq 0$ and $p \geq 1$ such that

$$|f(x) - f(x')| \leq \alpha\, |g(x) - g(x')|^p \quad \text{for all } x, x' \in [a, b].$$

Prove that if g is Riemann integrable, then f is also Riemann integrable.

29. Let $f : [0, 1] \to \mathbb{R}$ be a Riemann integrable function and let $p \geq 1$. Prove that the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left| f\left(\frac{j-1}{n}\right) - f\left(\frac{j}{n}\right) \right|^p$$

exists, and determine its value.

30. (Riemann's function) Prove that the function $f : [0, 1] \to \mathbb{R}$, defined by

$$f(x) := \begin{cases} 1 & \text{if } x = 0, \\ 1/q & \text{if } x \in [0,1] \cap \mathbb{Q},\ x = p/q,\ p, q \in \mathbb{N},\ \gcd(p, q) = 1, \\ 0 & \text{if } x \in [0, 1] \setminus \mathbb{Q}, \end{cases}$$

is Riemann integrable. (This function is also known in the literature as Thomae's function, the popcorn function, or the raindrop function.)

31. Prove that if the function $f : [a, b] \to \mathbb{R}$ has a finite limit at every point of $[a, b]$, then it is bounded.

32. Prove that the set of first-kind discontinuity points of a function $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, is at most countable.

33. Prove that if the function $f : [a, b] \to \mathbb{R}$ has a finite limit at every point of $[a, b]$, then it is Riemann integrable.

34. Prove that if $f : [0, 1] \to \mathbb{R}$ is a Riemann integrable function, then the function $g : [0, 1] \to \mathbb{R}$, defined by $g(x) = f(x^2)$, is also Riemann integrable.

35. Let $f : [0, 1] \to \mathbb{R}$ be a Riemann integrable function with the property that $0 < \left| \int_0^1 f(x)\, dx \right| \leq 1$. Prove that there exist $x_1, x_2 \in [0, 1]$ such that $x_1 \neq x_2$ and

$$\int_{x_1}^{x_2} f(x)\, dx = (x_1 - x_2)^{2002}.$$

36. Let $f:[0,1]\to\mathbb{R}$ be an increasing function and let $n\geq 1$ be an integer. Prove that

$$\int_0^1 f(x)\,dx \leq (n+1)\int_0^1 x^n f(x)\,dx.$$

Specify all continuous increasing functions f for which equality holds.

37. Calculate

$$\lim_{n\to\infty} n\int_0^{\infty} e^{-x}\sqrt[n]{e^x-1-\frac{x}{1!}-\frac{x^2}{2!}-\cdots-\frac{x^n}{n!}}\,dx.$$

38. Let $f:\left[0,\dfrac{\pi}{2}\right]\to\mathbb{R}$ be a continuous decreasing function. Prove that

$$\int_{\pi/2-1}^{\pi/2} f(x)\,dx \leq \int_0^{\pi/2} f(x)\cos x\,dx \leq \int_0^1 f(x)\,dx.$$

For which functions f do both inequalities hold with equality?

39. Let $f:[0,1]\to\mathbb{R}$ be a differentiable function and let $M\geq 0$ be such that $|f'(x)|\leq M$ for all $x\in[0,1]$. Prove that

$$\int_0^1 f^2(x)\,dx-\left(\int_0^1 f(x)\,dx\right)^2\leq\frac{M^2}{12}.$$

40. Let $f_1,\ldots,f_n:[0,1]\to(0,\infty)$ be continuous functions and let σ be a permutation of $\{1,\ldots,n\}$. Prove that

$$\prod_{i=1}^{n}\int_0^1\frac{f_i^2(x)}{f_{\sigma(i)}(x)}\,dx\geq\prod_{i=1}^{n}\int_0^1 f_i(x)\,dx.$$

41. a) Given continuous functions $u,v:[0,1]\to\mathbb{R}$, prove the Cauchy–Bunyakovsky–Schwarz inequality

$$\left(\int_0^1 u(x)v(x)\,dx\right)^2\leq\int_0^1 u^2(x)\,dx\int_0^1 v^2(x)\,dx.$$

b) Let $\mathcal{C}$ be the set of all differentiable functions $f:[0,1]\to\mathbb{R}$, with continuous derivative f' on $[0,1]$ and with $f(0)=0$, $f(1)=1$. Determine

$$\min_{f\in\mathcal{C}}\int_0^1\sqrt{1+x^2}\,(f'(x))^2\,dx,$$

as well as all functions $f\in\mathcal{C}$ for which this minimum is attained.

42. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that

$$\int_0^n f(x)f(n-x)\,dx = \int_0^n f^2(x)\,dx$$

for every natural number $n \geq 1$. Prove that f is periodic.

43. Let $f : [0, 1] \to \mathbb{R}$ be an integrable function such that

$$\int_0^1 f(x)\,dx = \int_0^1 xf(x)\,dx = 1.$$

Prove that $\displaystyle\int_0^1 f^2(x)\,dx \geq 4$.

44. Let $\mathcal{M}$ be the set of all continuous functions $f : [0, \pi] \to \mathbb{R}$, with the property that

$$\int_0^\pi f(x)\sin x\,dx = \int_0^\pi f(x)\cos x\,dx = 1.$$

Determine $\displaystyle\min_{f\in\mathcal{M}} \int_0^\pi f^2(x)\,dx$.

45. Let $p > 1$ be a real number and let $f : [a, b] \to [0, \infty)$ be a continuous function. Prove that

$$\left(\int_a^b f(x)\,dx\right)^p \leq (b-a)^{p-1}\int_a^b f(x)^p\,dx.$$

46. Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Prove that the function $\varphi : (0, \infty) \to \mathbb{R}$, defined by

$$\varphi(r) := \left(\frac{1}{b-a}\int_a^b |f(x)|^r\,dx\right)^{1/r},$$

is increasing.

47. Let $p > 1$ be a real number. For every continuous function $f : [0, 1] \to \mathbb{R}$ we denote

$$I(f) := \int_0^1 x^p|f(x)|\,dx - \int_0^1 x|f(x)|^p\,dx.$$

Determine the maximum value that $I(f)$ can take.

48. Let $f : [0,1] \to \mathbb{R}$ and $g : [0,1] \to (0,\infty)$ be continuous functions, with f increasing. Prove that for every $t \in [0,1]$ the inequality

$$\int_0^t f(x)g(x)\,dx \int_0^1 g(x)\,dx \le \int_0^t g(x)\,dx \int_0^1 f(x)g(x)\,dx$$

holds.

49. Let $f : [0,1] \to [0,\infty)$ be a monotone function.

(a) If f is decreasing, prove that

$$\int_0^1 f(x)\,dx \int_0^1 xf^2(x)\,dx \le \int_0^1 f^2(x)\,dx \int_0^1 xf(x)\,dx.$$

(b) If f is increasing, prove that

$$\int_0^1 xf^2(x)\,dx \int_0^1 f(x)\,dx \ge \int_0^1 f^2(x)\,dx \int_0^1 xf(x)\,dx.$$

50. Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable functions such that $m \le f(x) \le M$ for every $x \in [a,b]$ and $\displaystyle\int_a^b g(x)\,dx = 0$. Prove that

$$\left| \int_a^b f(x)g(x)\,dx \right| \le \frac{M-m}{2} \int_a^b |g(x)|\,dx.$$

51. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with continuous derivative on $[0,1]$. Prove that if $f(0) = 0$ and $0 < f'(x) \le 1$ for all $x \in [0,1]$, then

$$\left(\int_0^1 f(x)\,dx \right)^2 \ge \int_0^1 f^3(x)\,dx.$$

Give an example of a function f for which equality holds.

52. a) Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with continuous derivative on $[0,1]$. Prove that if there exists $a \in [0,1]$ such that $f(a) \le 0$, then

$$\int_0^1 f(t)\,dt \le \int_0^1 |f'(t)|\,dt.$$

b) Let $g : [0,1] \to \mathbb{R}$ be a differentiable function with continuous derivative on $[0,1]$. Prove that for every $x \in [0,1]$ the inequality

$$\int_0^1 \big(g(t) - |g'(t)|\big)\, dt \le g(x) \le \int_0^1 \big(g(t) + |g'(t)|\big)\, dt$$

holds.

53. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with continuous derivative on $[0,1]$. Prove that if $\displaystyle\int_0^1 f(x)\, dx = 0$, then for every $\alpha \in (0,1)$ the inequality

$$\left| \int_0^\alpha f(x)\, dx \right| \le \frac{1}{8} \max_{x\in[0,1]} |f'(x)|$$

holds.

54. Let $f : [-1,1] \to \mathbb{R}$ be a differentiable function, with the property that there exists $a \in (0,1)$ such that $\displaystyle\int_{-a}^a f(x)\, dx = 0$. Denoting $M := \sup_{x\in[-1,1]} |f'(x)|$, prove that:

a) $\displaystyle\left| \int_{-1}^1 f(x)\, dx \right| \le M(1-a)$.

b) $\displaystyle\int_{-1}^1 |f(x)|\, dx \ge 2|f(0)| - M$.

55. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function such that $f(0) = f(1) = 0$ and $|f'(x)| \le 1$ for all $x \in [0,1]$. Prove that

$$\left| \int_0^1 f(x)\, dx \right| \le \frac{1}{4}.$$

56. Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$, twice differentiable with bounded second derivative on (a,b), such that $f(a) = f(b) = 0$. Prove that

$$\int_a^b |f(x)|\, dx \le M \frac{(b-a)^3}{12}, \qquad \text{where } M := \sup\{|f''(x)| : x \in (a,b)\}.$$

57. Let $f : [-1,1] \to \mathbb{R}$ be a three-times differentiable function with continuous third derivative on $[-1,1]$, such that $f(-1) = f(0) = f(1) = 0$. Prove that

$$\int_{-1}^1 |f(x)|\, dx \le \frac{1}{12} \max_{x\in[-1,1]} |f'''(x)|.$$

58. (A. Beurling's inequality) Let $f : [a, b] \to \mathbb{R}$ be twice differentiable with continuous second derivative on $[a, b]$. Prove that if $f(a) = f(b) = 0$, then

$$\int_a^b |f''(x)|\, dx \geq \frac{4}{b-a} \max_{x\in[a,b]} |f(x)|.$$

59. Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function with continuous derivative on $[0, 1]$. Prove that if $f(0) = f(1) = -\dfrac{1}{6}$, then

$$\int_0^1 (f'(x))^2\, dx \geq 2 \int_0^1 f(x)\, dx + \frac{1}{4}.$$

60. Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function with continuous derivative on $[0, 1]$. Determine f, knowing that $f(1) = -\dfrac{1}{6}$ and

$$\int_0^1 (f'(x))^2\, dx \leq 2 \int_0^1 f(x)\, dx.$$

61. Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function with continuous derivative on $[0, 1]$. Prove that if $f\left(\dfrac{1}{2}\right) = 0$, then

$$\int_0^1 (f'(x))^2\, dx \geq 12 \left(\int_0^1 f(x)\, dx\right)^2.$$

62. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function with continuous derivative on $[a, b]$. Prove that if $f(a) = 0$, then

$$\int_a^b f(x)^2\, dx \leq (b-a)^2 \int_a^b f'(x)^2\, dx.$$

63. Let $f : [-1, 1] \to \mathbb{R}$ be a twice differentiable function with continuous second derivative on $[-1, 1]$. Prove that if $f(0) = 0$, then

$$\int_{-1}^1 (f''(x))^2\, dx \geq 10 \left(\int_{-1}^1 f(x)\, dx\right)^2.$$

64. Let $n \geq 0$ be an integer and let $f : [-1, 1] \to \mathbb{R}$ be $2n + 2$ times differentiable with continuous derivative of order $2n + 2$ on $[-1, 1]$. Prove that if $f(0) = f''(0) =$

$\cdots = f^{(2n)}(0) = 0$, then

$$\int_{-1}^{1} \left(f^{(2n+2)}(x)\right)^2 dx \geq (2n+2)!^2(4n+5)^2 \left(\int_{-1}^{1} f(x)\, dx\right)^2.$$

65. Prove that if $f : [0,1] \to \mathbb{R}$ is a convex function, then for every $n \in \mathbb{N}$ the inequality

$$\int_0^1 f(x)\, dx \leq \frac{1}{n+1}\left(f(0) + f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f(1)\right)$$

holds.

66. Let $f : [-1,1] \to \mathbb{R}$ be a continuous function, twice differentiable on $(-1,1)$, with the property that

$$2f'(x) + xf''(x) \geq 1 \quad \text{for all } x \in (-1,1).$$

Prove that $\displaystyle\int_{-1}^{1} xf(x)\, dx \geq \frac{1}{3}$.

67. Show that $(\cos t)^p \leq \cos(pt)$, for every $t \in \left[0, \frac{\pi}{2}\right]$ and $p \in (0,1)$.

68. Let $f : [0,1] \to \mathbb{R}$ be a function of class C^1, with $f(0) = 0$. Show that

$$\sup_{x\in[0,1]} |f(x)| \leq \sqrt{\int_0^1 [f'(x)]^2\, dx}.$$

69. Let $f : [1,+\infty) \to \mathbb{R}$ with $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + [f(x)]^2}.$$

Show that $\displaystyle\lim_{x\to\infty} f(x)$ exists and is smaller than $1 + \dfrac{\pi}{4}$.

70. Let f and $g : [0,1] \to (0,+\infty)$ be two continuous functions satisfying the relation

$$\sup_{x\in[0,1]} f(x) = \sup_{x\in[0,1]} g(x).$$

Show that there exists $t \in [0,1]$ with $[f(t)]^2 + 3f(t) = [g(t)]^2 + 3g(t)$.

71. Let $y(h) = 1 - 2\sin^2(2\pi h)$ and $f(y) = \dfrac{2}{1 + \sqrt{1-y}}$. Show that

$$f(y(h)) = 2 - 4\sqrt{2\pi}\,|h| + O(h^2),$$

where $\limsup_{h\to 0} \dfrac{O(h^2)}{h^2} < \infty$.

72. Let $f : [0,1] \to \mathbb{R}^+$ be a continuous function. Suppose there exist positive constants m and M such that $(f(x) - m)(f(x) - M) \leq 0$ for all $x \in [0,1]$. Prove that

$$\int_0^1 f(x)\,dx \int_0^1 \frac{1}{f(x)}\,dx \leq \frac{(M+m)^2}{4Mm}.$$

73. Let f be a non-negative Riemann integrable function on the interval $[0,1]$ satisfying $\displaystyle\int_0^1 f(x)\,dx = 1$. Prove the inequality

$$\int_0^1 \left(x - \int_0^1 uf(u)\,du\right)^2 f(x)\,dx \leq \frac{1}{4}.$$

74. Let $f : [1,2] \to \mathbb{R}$ be a differentiable function such that its derivative satisfies $|f'(x)| \leq 1$ for all $x \in [1,2]$. Prove the inequality

$$\left|\int_1^2 f(x)\,dx - \frac{f(1)+f(2)}{2}\right| \leq \frac{1 - [f(1) - f(2)]^2}{4}.$$

75. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying the Lipschitz condition $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$, where $M > 0$ is a constant. Prove that for any real numbers $a < b$:

(i) $\displaystyle\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq M\frac{b-a}{4}$.

(ii) $\displaystyle\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}\right| \leq M\frac{b-a}{3}$.

76. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Show that

$$\left|f(1) - \int_0^1 f(x)\,dx\right| \leq \frac{1}{2}\max_{x\in[0,1]} |f'(x)|.$$

77. Find the smallest constant C such that for every real polynomial $P(x)$ of degree 3 that has a root in the interval $[0, 1]$,

$$\int_0^1 |P(x)|\, dx \leq C \max_{x\in[0,1]} |P(x)|.$$

78. Let $f : [0, 1] \to [0, \infty)$ be a non-identically-zero, differentiable function with continuous derivative.

a) Prove that the sequence $a_n = \displaystyle\int_0^1 \frac{f(x)}{x^n + 1}\, dx$ $(n \in \mathbb{N}^*)$ is convergent.

b) If $a = \lim_{n\to\infty} a_n$, determine $\displaystyle\lim_{n\to\infty} \frac{a_n}{a^n}$.

79. Let $f : [0, 1] \to [0, 1]$ be an increasing function and let

$$a_n = \int_0^1 \frac{1 + f^n(x)}{1 + f^{n+1}(x)}\, dx \qquad (n \geq 1).$$

Prove that the sequence $(a_n)_{n\geq 1}$ is convergent and determine its limit.

80. a) Given an integer $n \geq 0$, calculate $\displaystyle\int_0^1 (1 - t)^n e^t\, dt$.

b) Let $k \geq 0$ be an integer and let $(x_n)_{n\geq k}$ be the sequence with general term

$$x_n = \sum_{i=k}^{n} \binom{i}{k} \left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{i!}\right).$$

Prove that $(x_n)_{n\geq k}$ is convergent and determine its limit.

81. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function and let $(a_n), (b_n)$ be sequences of real numbers with the property that

$$\lim_{n\to\infty} \int_0^1 |f(x) - a_n x - b_n|\, dx = 0.$$

Prove that:

a) The sequences (a_n) and (b_n) are convergent.

b) There exist $a, b \in \mathbb{R}$ such that $f(x) = ax + b$ for all $x \in \mathbb{R}$.

82. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function with the property that

$$\int_0^1 f(x)g(x)\, dx = \int_0^1 f(x)\, dx \int_0^1 g(x)\, dx$$

for every continuous, non-differentiable function $g : [0,1] \to \mathbb{R}$. Prove that f is constant.

83. a) Prove that $\ln(1+x) \le x$ for all $x \ge 0$.

b) If $a > 0$, prove that $\lim_{n\to\infty} n \int_0^1 \frac{x^n}{a+x^n}\,dx = \ln \frac{a+1}{a}$.

84. Let $f : [0,1] \to \mathbb{R}$ be the function $f(x) = (x^2+1)e^x$. Calculate

$$\lim_{n\to\infty} n \int_0^1 \left(f\left(\frac{x}{2^n}\right) - 1 \right) dx.$$

85. For each $n \in \mathbb{N}^*$ consider the function $f_n : [0,n] \to \mathbb{R}$, defined by $f_n(x) = \arctan([x])$, where $[x]$ denotes the integer part of x. Show that f_n is Riemann integrable and determine

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n f_n(x)\,dx.$$

86. For each $\alpha \in (0,1]$ denote

$$I_n(\alpha) = \int_0^\alpha \ln\left(1 + x + x^2 + \cdots + x^{n-1}\right) dx, \qquad n \ge 2.$$

Calculate:
a) $\lim\limits_{n\to\infty} I_n(\alpha)$ for $\alpha \in (0,1)$;
b) $\lim\limits_{n\to\infty} I_n(1)$.

87. Let $\alpha > 0$ and let $a, b \in \mathbb{R}$ be such that $a < b$. Determine

$$\lim_{n\to\infty} \int_a^b \sqrt[n]{(x-a)^n + (b-x)^n + \alpha\big((x-a)(b-x)\big)^{n/2}}\,dx.$$

88. Let $\{\epsilon_n\}_{n=1}^\infty$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \epsilon_n = 0$. Evaluate the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \ln\left(\frac{k}{n} + \epsilon_n\right).$$

89. Let α, β be reals such that $2\beta \geq \alpha$ and suppose that they are not simultaneously negative. Let

$$F(\alpha, \beta) = \lim_{n\to\infty} n^{\alpha} \int_0^{n^{-\beta}} x^{x+1}\, dx.$$

Prove that $F(\alpha, \beta)$ exists and is finite. Hence find the value of $F(\alpha, \beta)$.

90. Let $\psi : \mathbb{R} \to \mathbb{R}$ be such that

$$\int_x^{x+y} \psi(t)\, dt = \int_{x-y}^{x} \psi(t)\, dt.$$

Find all possible $\psi(x)$.

91. (Wirtinger's inequalities)
a) Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function with continuous derivative on $[0, 1]$, with the property that $\int_0^1 f(x)\, dx = 0$. Prove that

$$\int_0^1 f(x)^2\, dx \leq \frac{1}{2} \int_0^1 f'(x)^2\, dx.$$

b) Let $f : [0, 1] \to \mathbb{R}$ be differentiable and satisfy $f(0) = 0$. Prove that

$$\int_0^1 [f'(x)]^2\, dx \geq 3 \left(\int_0^1 f(x)\, dx \right)^2.$$

c) Let $f : [0, 1] \to \mathbb{R}$ be differentiable and satisfy $f(0) = f(1) = 0$. Prove that

$$\int_0^1 [f'(x)]^2\, dx \geq 12 \left(\int_0^1 f(x)\, dx \right)^2.$$

d) Let $f : [0, 1] \to \mathbb{R}$ be differentiable and satisfy $f(0) = f(1) = 0$. Prove that

$$\int_0^1 [f'(x)]^2\, dx \geq \pi^2 \int_0^1 [f(x)]^2\, dx.$$

e) Let $f : [0, 1] \to \mathbb{R}$ be twice continuously differentiable and satisfy $f(0) = f'(0) = 0$. Prove that

$$\int_0^1 [f''(x)]^2\, dx \geq 20 \left(\int_0^1 f(x)\, dx \right)^2.$$

f) Let $f : [0, 1] \to \mathbb{R}$ be twice continuously differentiable and satisfy $f(0) = f(1) = 0$, $f'(0) = f'(1) = 0$. Prove that

$$\int_0^1 [f''(x)]^2\, dx \geq 720 \left(\int_0^1 f(x)\, dx \right)^2.$$

g) Let $f : [0,1] \to \mathbb{R}$ be twice continuously differentiable and satisfy $f(0) = f'(0) = 0$. Prove that

$$\int_0^1 [f''(x)]^2\,dx \geq \frac{45}{2}\left(\int_0^1 xf(x)\,dx\right)^2.$$

h) Let $f : [0,1] \to \mathbb{R}$ be differentiable and satisfy $f(0) = f(1) = 0$. Find the smallest constant C such that

$$\max_{x\in[0,1]} |f(x)| \leq C\sqrt{\int_0^1 [f'(t)]^2\,dt}.$$

i) Let $f : [0,1] \to \mathbb{R}$ be differentiable with $f(0) = f(1) = 0$. Find the best constant C such that

$$|f(c)|^2 \leq C\int_0^1 [f'(x)]^2\,dx, \qquad \forall c \in [0,1].$$

j) Let $f : [0,1] \to \mathbb{R}$ be twice continuously differentiable and satisfy $f(0) = f(1) = 0$. Find the best constant C such that

$$\int_0^1 |f''(x)|\,dx \geq C \max_{x\in[0,1]} |f(x)|.$$

k) Let $f : [0,1] \to \mathbb{R}$ be twice continuously differentiable, $f(0) = 0$, $f'(0) = 0$. Find the best constant C such that

$$f(1)^2 \leq C\int_0^1 [f''(x)]^2\,dx.$$

l) Let $M = \{f \mid f \in C^2[0,1],\ f(0) = f(1) = 0,\ f'(0) = 1\}$. Determine

$$\min_{f\in M} \int_0^1 (f''(x))^2\,dx$$

and the functions for which the minimum is achieved.

m) Let $\mathcal{F}$ be the set of continuously differentiable functions $f : [0,1] \to \mathbb{R}$ such that $\displaystyle\int_0^1 f(x)\,dx = 0$, $f \not\equiv 0$. Find the largest constant K such that

$$\int_0^1 [f'(x)]^2\,dx \geq K\left(\int_0^1 x^2 f(x)\,dx\right)^2,$$

and determine the case of equality.

92. Let $M > 0$ be a real number and let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function with continuous second derivative on $[a,b]$, with the property that $|f''(x)| \leq M$ for all

$x \in [a, b]$. Prove that

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f\left(\frac{a+b}{2}\right) \right| \leq M \frac{(b-a)^2}{24}.$$

93. Let $a \in [0, 1]$. Determine all closed intervals $I \subset [0, 1]$ with the property that for every differentiable convex function $f : [0, 1] \to \mathbb{R}$ the inequality

$$\int_0^1 \max\{f(a), f(x)\}\, dx \geq \max_{x \in I} f(x)$$

holds.

94. Let $f_0 : [0, 1] \to \mathbb{R}$ be a continuous function, and let f_{n+1} be the antiderivative of f_n on $[0, 1]$ that vanishes at the origin, for $n \in \mathbb{N}$. If $f_{1982}(1) = \dfrac{1}{1983!}$, prove that there exists $x_0 \in [0, 1]$ such that $f_0(x_0) = x_0$.

95. Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Prove that:

1) for each $\alpha \in (0, 1)$ there exists at least one $c \in (a, b)$ such that $\alpha \displaystyle\int_a^b f(x)\, dx = \int_a^c f(x)\, dx$;

2) if f is continuous on (a, b) and $\displaystyle\int_a^b f(x)\, dx \neq 0$, then for every $n \in \mathbb{N}^*$ there exist n distinct points $x_1, x_2, \ldots, x_n$ in the interval (a, b) such that

$$\int_a^b f(x)\, dx = \frac{n(b-a)}{\dfrac{1}{f(x_1)} + \dfrac{1}{f(x_2)} + \cdots + \dfrac{1}{f(x_n)}}.$$

96. Prove that

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m(m+1)\cdots(m+n)} = \int_{0+0}^{1} \frac{e^x - 1}{x}\, dx.$$

97. (Generalization of the preceding problem) Prove that for every $m \in \mathbb{N}$ the equality

$$\sum_{n_m=1}^{\infty} \cdots \sum_{n_1=1}^{\infty} \sum_{n_0=1}^{\infty} \frac{1}{n_0(n_0+n_1)\cdots(n_0+n_1+\cdots+n_m)}$$

$$= (-1)^{m-1}\left(\int_{0+0}^{1} \frac{e^x - 1}{x}\, dx + \sum_{j=1}^{m-1} \frac{1}{j}\left(1 - e\sum_{i=0}^{j-1} \frac{(-1)^i}{i!}\right)\right)$$

holds.

98. Determine the continuous decreasing functions $f : [0,1] \to [0,\infty)$ that simultaneously satisfy the conditions

$$\int_0^1 f(x)\, dx \le 1 \qquad \text{and} \qquad \int_0^1 f(x^3)\, dx \le 3\left(\int_0^1 x f(x)\, dx\right)^2.$$

99. Define $g_0 = 1$ and $g_{n+1}(x) = 1 + \displaystyle\int_0^x g_n(t - t^2)\, dt$ for $x \in [0,1]$.

a) Show that $0 \le g_n(x) - g_{n-1}(x) \le \dfrac{x^n}{n!}$.
b) Deduce the pointwise convergence of (g_n).
c) Prove that the convergence is uniform on $[0,1]$.

100. Let $f \in C[a,b]$ be such that $\displaystyle\int_a^b x^n f(x)\, dx = 0$ for every $n \in \mathbb{N}$. Show that f is identically zero.

101. Let $f \in C^1[a,b]$ with $f'(a) \ne 0$. For any $x \in (a,b]$, let $\theta(x) \in [a,x]$ be such that

$$\int_a^x f(t)\, dt = (x-a) f(\theta(x)).$$

Calculate $\displaystyle\lim_{x\to a} \frac{\theta(x) - a}{x - a}$.

102. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(x)\, dx = \int_0^1 x f(x)\, dx = \cdots = \int_0^1 x^{n-1} f(x)\, dx = 0 \quad \text{and} \quad \int_0^1 x^n f(x)\, dx = 1.$$

Show that there exists $a \in [0,1]$ such that $|f(a)| \ge 2^{n(n+1)}$.

103. Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function (i.e., $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$) that is integrable on every compact interval of $\mathbb{R}$. Show that there exists $a \in \mathbb{R}$ such that $f(x) = ax$ for all $x \in \mathbb{R}$.

104. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Show that

$$\int_0^1 \int_x^1 f(t)\,dt\,dx = \int_0^1 x f(x)\,dx.$$

105. Let $f_1, f_2, f_3, f_4 \in \mathbb{R}[x]$. Show that

$$F(x) = \int_1^x f_1(t)f_2(t)\,dt \int_1^x f_3(t)f_4(t)\,dt - \int_1^x f_1(t)f_4(t)\,dt \int_1^x f_2(t)f_3(t)\,dt$$

is a polynomial divisible by $(x-1)^4$.

106. Let $f : [0, \pi] \to \mathbb{R}$ be a continuous function such that

$$\int_0^\pi f(x)\cos x\,dx = \int_0^\pi f(x)\sin x\,dx = 0.$$

Show that the equation $f(x) = 0$ has at least two roots in $(0, \pi)$.

107. Let $f : [-1, 1] \to \mathbb{R}$ be a twice differentiable function such that $|f(x)| \leq 1$ and $|f''(x)| \leq 1$ for all $x \in [-1, 1]$. Show that $|f'(x)| \leq 2$ for all $x \in [-1, 1]$.

108. Determine the maximum of the expression

$$\int_0^1 x^2 f(x)\,dx - \int_0^1 x(f(x))^2\,dx$$

for $f \in C[0, 1]$.

109. Let $I \subseteq \mathbb{R}$ be an open interval with $0 \in I$ and $f : I \to \mathbb{R}$ a continuous function with $f(0) = 0$, right-differentiable at 0.

a) Show that $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) = \frac{1}{2} f'_d(0)$.

b) Special case: calculate $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \left(\sqrt{1 + \frac{k}{n^2}} - 1\right)$.

110. Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function. Assume there exists $\displaystyle\lim_{x\to\infty} (f(x) + f'(x)) = l \in \mathbb{R}$. Show that

$$\lim_{x\to\infty} f(x) = l \qquad \text{and} \qquad \lim_{x\to\infty} f'(x) = 0.$$

111. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function. If $f(x+y)f(x-y) \leq f^2(x)$ for all $x, y \in \mathbb{R}$, show that $f(x)f''(x) \leq (f'(x))^2$ for all $x \in \mathbb{R}$.

112. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ an arbitrary function. Show that f has at most a countable set of strict extrema.

113. Suppose the differentiable functions $a, b, f, g : \mathbb{R} \to \mathbb{R}$ satisfy $f(x) \geq 0$, $f'(x) \geq 0$, $g(x) > 0$, $g'(x) > 0$ for all $x \in \mathbb{R}$. Let $\lim\limits_{x\to\infty} a(x) = A > 0$, $\lim\limits_{x\to\infty} b(x) = B > 0$, $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} g(x) = \infty$, and assume

$$\lim_{x\to\infty} \left(\frac{f'(x)}{g'(x)} + a(x)\frac{f(x)}{g(x)} - b(x) \right) = 0.$$

Show that $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \dfrac{B}{A+1}$.

114. Let $k > 1$ be a real number. Calculate the limits:

a) $L = \lim\limits_{n\to\infty} \displaystyle\int_0^1 \sqrt[n]{kx+k-1}\, dx$;

b) $\lim\limits_{n\to\infty} n \left(L - \displaystyle\int_0^1 \sqrt[n]{kx+k-1}\, dx \right)$.

115. Let $f : [0,1] \to \mathbb{R}$ be the function $f(x) = (x^2+1)e^x$. Calculate

$$\lim_{n\to\infty} \int_0^1 \left(f\left(\frac{x}{n}\right) \right)^n dx.$$

116. Assume $f(x) \geq 0$ and $\displaystyle\int_0^\infty f^2(x)\, dx < \infty$. Define $F(x) = \displaystyle\int_0^x f(t)\, dt$. Prove that

$$\int_0^\infty \left(\frac{F(x)}{x} \right)^2 dx \leq 4 \int_0^\infty f^2(x)\, dx.$$

117. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable, satisfying $f(0) = 0$, $f(1) = 1$, and $f'(x) > 0$. Define

$$L = \int_0^1 \sqrt{1 + [f'(x)]^2}\, dx.$$

Prove that

$$\int_0^1 x\sqrt{1 + [f'(x)]^2}\, dx \leq \frac{L}{2} + \frac{1}{4}.$$

118. (Gronwall's inequality) Let $u(t)$ be a nonnegative continuous function on $[a, b]$. Assume there exist constants $C \geq 0$, $K \geq 0$ such that

$$u(t) \leq C + K \int_a^t u(s)\, ds, \qquad \forall t \in [a, b].$$

Prove that $u(t) \leq Ce^{K(t-a)}$.

119. Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove that there exists $c \in (a, b)$ such that

$$f(c) \left(\int_c^b f(x)\, dx - \int_a^c f(x)\, dx \right) = 0.$$

120. Let $f : [0, 1] \to \mathbb{R}$ be continuous and positive. Prove that there exists $c \in (0, 1)$ such that

$$\int_0^c f(x)\, dx = (1 - c) f(c).$$

121. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Prove that for every positive integer n, there exists $c \in (0, 1)$ such that

$$\int_0^c x^{n-1} f(x)\, dx = \frac{c^n}{n} f(c).$$

122. Let $f : [0, \frac{\pi}{2}] \to \mathbb{R}$ be continuous and satisfy

$$\int_0^{\pi/2} f(x) \sin x\, dx = \int_0^{\pi/2} f(x) \cos x\, dx.$$

Prove that there exists $c \in (0, \frac{\pi}{2})$ such that

$$f(c) = \left(\int_0^c f(x)\, dx \right) (\sin c + \cos c).$$

123. Let $f : [0, 1] \to \mathbb{R}$ be differentiable and $k \neq 0$. Assume $f(0) = 0$ and

$$f(1) = k \int_0^1 e^{k(1-x)} f(x)\, dx.$$

Prove that there exists $c \in (0, 1)$ such that $f'(c) = kf(c)$.

124. Let $f : [a, b] \to \mathbb{R}$ be continuous and assume $\displaystyle\int_a^b f(x)\,dx \neq 0$. Prove that there exists $c \in (a, b)$ such that

$$f(c)\left[\frac{1}{\int_a^c f(x)\,dx} - \frac{1}{\int_c^b f(x)\,dx}\right] = \frac{1}{c-a} - \frac{1}{b-c}.$$

125. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Prove that there exists $c \in (0, 1)$ such that

$$2c^2 f(c)\left(\int_0^c f(x)\,dx - \int_c^1 f(x)\,dx\right) = \int_0^c f(x)\,dx \int_c^1 f(x)\,dx.$$

126. Let $f : [0, 1] \to \mathbb{R}$ be continuously differentiable, $f(0) = 0$, $f(1) = 1$, and $f'(x) + f(x) > 0$. Find the minimum value of

$$I = \int_0^1 \frac{1}{f'(x) + f(x)}\,dx.$$

127. Under the same hypotheses as the preceding problem, find the minimum value of

$$J = \int_0^1 \frac{1}{[f'(x) + f(x)]^2}\,dx.$$

128. Let f be continuous on $[0, 1]$ and satisfy $1 \le f(x) \le 3$. Prove that

$$\int_0^1 f(x)\,dx \int_0^1 \frac{1}{f(x)}\,dx \le \frac{4}{3}.$$

129. (Polya–Szegő inequality) Let f, g be positive integrable functions on $[0, 1]$ and assume

$$0 < m \le \frac{f(x)}{g(x)} \le M.$$

Prove that

$$\int_0^1 f^2(x)\,dx \int_0^1 g^2(x)\,dx \le \frac{(m+M)^2}{4mM}\left(\int_0^1 f(x)g(x)\,dx\right)^2.$$

130. Let f be continuous on $[0, 1]$ and satisfy $1 \le f(x) \le 3$. Find the maximum value of

$$P = \int_0^1 f(x)\,dx + \int_0^1 \frac{1}{f(x)}\,dx - \int_0^1 f(x)\,dx \int_0^1 \frac{1}{f(x)}\,dx.$$

131. Let $f : [0, 1] \to \mathbb{R}$ be continuously differentiable and satisfy $1 \le f(x) \le 3$. Prove that

$$\int_0^1 f(x)\,dx \int_0^1 \frac{1}{f(x)}\,dx \le \min\left\{\frac{4}{3},\ 1 + \frac{1}{\pi^2}\int_0^1 \left(\frac{f'(x)}{f(x)}\right)^2 dx\right\}.$$

132. Let $f, g : \mathbb{R} \to \mathbb{R}$ be nonconstant, differentiable functions satisfying

$$f(x+y) = f(x)f(y) - g(x)g(y), \qquad g(x+y) = f(x)g(y) + g(x)f(y),$$

for all $x, y \in \mathbb{R}$, and $f'(0) = 0$. Prove that $(f(x))^2 + (g(x))^2 = 1$ for all $x \in \mathbb{R}$.

133. Let f be a real function on the real line with a continuous third derivative. Prove that there exists a point $a \in \mathbb{R}$ such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \ge 0.$$

134. Suppose that $f : [0, 1] \to \mathbb{R}$ has a continuous derivative and that $\displaystyle\int_0^1 f(x)\,dx = 0$. Prove that for every $\alpha \in (0, 1)$,

$$\left|\int_0^\alpha f(x)\,dx\right| \le \frac{1}{8} \max_{0 \le x \le 1} |f'(x)|.$$

135. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function satisfying $xf(y) + yf(x) \le 1$ for every $x, y \in [0, 1]$.
a) Show that $\displaystyle\int_0^1 f(x)\,dx \le \frac{\pi}{4}$.
b) Find such a function for which equality occurs.

136. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. A point x is called a *shadow point* if there exists a point $y \in \mathbb{R}$ with $y > x$ such that $f(y) > f(x)$. Let $a < b$ be real numbers and suppose that all the points of the open interval $I = (a, b)$ are shadow points, while a and b are not shadow points. Prove that:
a) $f(x) \le f(b)$ for all $a < x < b$;
b) $f(a) = f(b)$.

137. Let $(a_n) \subset \left[\frac{1}{2}, 1\right]$. Define the sequence $x_0 = 0$, and

$$x_{n+1} = \frac{a_{n+1} + x_n}{1 + a_{n+1}x_n}.$$

Is this sequence convergent? If yes, find its limit.

138. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$. Prove that if for any real x the sequence $x, f(x), f(f(x)), \ldots$ is an arithmetic progression, then there is $a \in \mathbb{R}$ such that $f(x) = x + a$ for all $x \in \mathbb{R}$.

139. Let $f : [-1, 1] \to \mathbb{R}$ be a continuous function having finite derivative at 0, and let

$$I(h) = \int_{-h}^{h} f(x)\, dx, \qquad h \in [0, 1].$$

Prove that:
a) There exists $M > 0$ such that $|I(h) - 2f(0)h| \leq Mh^2$ for any $h \in [0, 1]$.
b) The sequence $(a_n)_{n\geq 1}$ defined by $a_n = \sum_{k=1}^{n} \sqrt{k} \left| I\left(\frac{1}{k}\right) \right|$ is convergent if and only if $f(0) = 0$.

140. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, two times derivable on $\mathbb{R}$, for which there exists $c \in \mathbb{R}$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for all $a \neq b \in \mathbb{R}$. Prove that $f''(c) = 0$.

141. a) Consider a differentiable and convex function $f : [0, \infty) \to [0, \infty)$. Show that if $f(x) \leq x$ for every $x \geq 0$, then $f'(x) \leq 1$ for every $x \geq 0$.
b) Determine differentiable and convex functions $f : [0, \infty) \to [0, \infty)$ which have the property that $f(0) = 0$ and $f'(x)f(f(x)) = x$ for every $x \geq 0$.

142. Let $f : \mathbb{R} \to [0, \infty)$ be a twice differentiable function whose second derivative is continuous, such that

$$\lim_{x\to\infty} \left[f(x) + f'(x) + f''(x)\right] = C.$$

Prove that $\lim_{x\to\infty} f(x) = C$.

143. Let f be a real function with a continuous third derivative such that $f(x)$, $f'(x)$, $f''(x)$, and $f'''(x)$ are positive for all x. Suppose that $f'''(x) \leq f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x.

144. Suppose that f is a function on the interval $[1, 3]$ such that $-1 \leq f(x) \leq 1$ for all x and $\int_{1}^{3} f(x)\, dx = 0$. How large can $\int_{1}^{3} \frac{f(x)}{x}\, dx$ be?

145. a) Prove that there exists a differentiable function $f : (0, \infty) \to (0, \infty)$ such that $f(f'(x)) = x$ for all $x > 0$.
b) Prove that there is no differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f'(x)) = x$ for all $x \in \mathbb{R}$.

146. Let $a > 0$ and let $\mathcal{F} = \{f : [0, 1] \to \mathbb{R} \mid f \text{ is concave and } f(0) = 1\}$. Determine

$$\min_{f\in\mathcal{F}} \left\{ \left(\int_0^1 f(x)\,dx \right)^2 - (a + 1) \int_0^1 x^{2a} f(x)\,dx \right\}.$$

147. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that there exists $M > 0$ satisfying

$$|f(x_1 + \cdots + x_n) - f(x_1) - \cdots - f(x_n)| \le M, \qquad \forall x_1, \ldots, x_n \in \mathbb{R}.$$

Prove that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

148. Let $f : (0, 1) \to \mathbb{R}$ be a function satisfying $\lim\limits_{x\to 0^+} f(x) = 0$. If there exists $\lambda \in (0, 1)$ such that $\lim\limits_{x\to 0^+} \dfrac{|f(x) - f(\lambda x)|}{x} = 0$, prove that

$$\lim_{x\to 0^+} \frac{f(x)}{x} = 0.$$

149. Determine twice differentiable functions $f : \mathbb{R} \to \mathbb{R}$ which verify the relation

$$(f'(x))^2 + f''(x) \le 0, \qquad \forall x \in \mathbb{R}.$$

150. For an arbitrary number $x_0 \in (0, \pi)$, define recursively the sequence $(x_n)_{n\ge 0}$ by $x_{n+1} = \sin(x_n)$. Compute

$$\lim_{n\to\infty} \sqrt{n}\, x_n.$$

151. Let a be a real number, and $f : \mathbb{R} \to \mathbb{R}$ a non-decreasing function. Prove that f is continuous at a if and only if there exists a sequence $(a_n)_{n\ge 1}$ of positive real numbers such that

$$\int_a^{a+a_n} f(x)\,dx + \int_a^{a-a_n} f(x)\,dx \le \frac{a_n}{n}$$

for all natural numbers n.

152. Let $f : \mathbb{R} \to \mathbb{R}$ be a surjective function that has the property that if the sequence $(f(x_n))_{n\geq 1}$ is convergent, then the sequence $(x_n)_{n\geq 1}$ is convergent. Prove that f is continuous.

153. Let $f : [0, \infty) \to (0, \infty)$ be a continuous function.
a) Show that there exists a natural number n_0 and a sequence of positive real numbers $(x_n)_{n\geq n_0}$ satisfying

$$n \int_0^{x_n} f(t)\, dt = 1, \qquad \forall n > n_0.$$

b) Prove that the sequence $(nx_n)_{n\geq n_0}$ is convergent and find its limit.

154. Let $f : [a, b] \to \mathbb{R}$ be a function with the Intermediate Value Property such that $f(a) * f(b) < 0$. Show that there exist α, β such that $a < \alpha < \beta < b$ and

$$f(\alpha) + f(\beta) = f(\alpha) * f(\beta).$$

155. Let $f : [0, 1] \to (0, 1)$ be a surjective function.
a) Prove that f has at least one point of discontinuity.
b) Given that f admits a limit at any point of the interval $[0, 1]$, show that it has at least two points of discontinuity.

156. Let $f : [0, \infty) \to (0, \infty)$ be an increasing function and $g : [0, \infty) \to \mathbb{R}$ a twice differentiable function such that g'' is continuous and

$$g''(x) + f(x)g(x) = 0, \qquad \forall x \geq 0.$$

a) Provide an example of such functions, with $g \not\equiv 0$.
b) Prove that g is bounded.

157. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function, constant on $\mathbb{Z}_{\geq 0}$. For any $0 \leq a < b < c < d$ which satisfy $f(a) = f(c)$ and $f(b) = f(d)$, we also have

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{c+d}{2}\right).$$

Prove that f is a constant function.

158. Let f_n be a sequence of real-valued C^1 functions on $[0, 1]$ such that for all $n \in \mathbb{N}$ the following hold:

$$|f_n'(x)| \leq \frac{1}{\sqrt{x}} \quad (0 < x \leq 1), \qquad \int_0^1 f_n(x)\, dx = 0.$$

Prove that f_n has a convergent subsequence that converges uniformly on $[0, 1]$.

159. Let $\chi_{\mathbb{Q}}$ denote the characteristic function of the rationals in $[0,1]$. Does there exist a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ such that f_n converges to $\chi_{\mathbb{Q}}$ pointwise?

160. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(t)\,dt = \int_0^1 tf(t)\,dt.$$

Prove that there exists $c \in (0,1)$ such that

$$\int_0^c f(t)\,dt = \frac{c}{2} f(c).$$

161. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $\displaystyle\int_0^1 f(t)\,dt = \int_0^1 tf(t)\,dt.$ Prove that there exists $c \in (0,1)$ such that

$$cf(c) = 2\int_c^1 f(t)\,dt.$$

162. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$f'(x) = f^2(x) f(-x).$$

Find an explicit formula for f.

163. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $\displaystyle\int_0^1 f(t)\,dt = 1$ and

$$\int_0^1 (1 - f(x)) e^{-f(x)}\,dx \leq 0.$$

Prove that $f(x) = 1$ for all $x \in [0,1]$.

164. Let $f : [a,b] \to [0,+\infty)$ be a continuous and not everywhere 0 function. Prove that

$$\lim_{n \to +\infty} \frac{\displaystyle\int_a^b f^{n+1}(t)\,dt}{\displaystyle\int_a^b f^n(t)\,dt} = \sup_{x \in [a,b]} f(x).$$

165. Examine if there exists a continuous function $f : [1, +\infty) \to \mathbb{R}$ such that $f(x) > 0$ for all $x \in [1, +\infty)$ and $\displaystyle\int_1^\infty f(t)\,dt$ converges whereas $\displaystyle\int_1^\infty f^2(t)\,dt$ diverges.

166. Let $0 < x_1 < 1$ and $x_{n+1} = x_n(1 - x_n)$ for $n = 1, 2, 3, \ldots$. Show that

$$\lim_{n\to\infty} n x_n = 1.$$

167. a) Let $(x_n)_{n\geq 1}$ be a sequence such that $x_1 \in (0, 1)$ and $x_{n+1} = x_n(1 - x_n^2)$ for $n \geq 1$. Evaluate

$$\lim_{n\to\infty} \sqrt{n}\, x_n.$$

b) Let $(x_n)_{n\geq 1}$ be a sequence such that $x_1 = x > 1$ and $x_{n+1} = x_n + \sqrt{x_n - 1}$ for $n \geq 1$. Evaluate

$$\lim_{n\to\infty} \frac{4x_n - n^2}{n \log n}.$$

168. Evaluate the following limits:

a) $\displaystyle\lim_{n\to\infty} \left(\frac{\left(1 + \frac{1}{n}\right)^n}{e} \right)^n$;

b) $\displaystyle\lim_{n\to\infty} \prod_{k=1}^{n} \left(\frac{k}{k+n} \right)^{e^{1999/n} - 1}$.

169. Define the sequence $x_1, x_2, \ldots$ inductively by $x_1 = \sqrt{5}$ and $x_{n+1} = x_n^2 - 2$ for each $n \geq 1$. Compute

$$\lim_{n\to\infty} \frac{x_1 x_2 \cdots x_n}{x_{n+1}}.$$

170. Suppose that $a_0 = 1$ and that $a_{n+1} = a_n + e^{-a_n}$ for $n = 0, 1, 2, \ldots$. Does the sequence $(a_n - \log n)_{n\geq 1}$ have a finite limit?

171. Let k be an integer greater than 1. Suppose that $a_0 > 0$, and define the sequence $(a_n)_{n\geq 0}$ by

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}, \qquad n \geq 0.$$

Evaluate

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k}.$$

172. a) Does there exist a sequence $(a_n)_{n\geq 1}$ of positive reals such that the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{a_n}$ converges, and $\displaystyle\prod_{k=1}^{n} a_k \leq n^n$ for all $n \geq 1$?

b) Does there exist a sequence $(a_n)_{n\geq 1}$ of positive reals such that $\displaystyle\sum_{k=1}^{n}\frac{1}{a_k} \leq 2008$, and $\displaystyle\sum_{k=1}^{n} a_k \leq n^2$ for all $n \geq 1$?

173. Let $(a_n)_{n\geq 1}$ be a decreasing sequence of positive reals. Let $s_n = a_1 + a_2 + \cdots + a_n$ and

$$b_n = \frac{1}{a_{n+1}} - \frac{1}{a_n}, \qquad n \geq 1.$$

Prove that if $(s_n)_{n\geq 1}$ is convergent, then $(b_n)_{n\geq 1}$ is unbounded.

174. Let $(x_n)_{n\geq 2}$ be a sequence of real numbers such that $x_2 > 0$ and

$$x_{n+1} = -1 + \sqrt[n]{1 + n x_n}, \qquad n \geq 2.$$

Show that:
a) $\displaystyle\lim_{n\to\infty} x_n = 0$;
b) $\displaystyle\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$ and $\displaystyle\lim_{n\to\infty} n x_n = 0$.

175. Let $(a_n)_{n\geq 1}$ be a sequence of positive reals. Show that

$$\limsup_{n\to\infty} n\left(\frac{1 + a_{n+1}}{a_n} - 1\right) \geq 1,$$

and

$$\limsup_{n\to\infty} \left(\frac{a_1 + a_{n+1}}{a_n}\right)^n \geq e.$$

176. Let $(a_n)_{n\geq 1}$ be a sequence such that $x_n = \displaystyle\sum_{k=1}^{n} a_k^2$ converges, while $y_n = \displaystyle\sum_{k=1}^{n} a_k$ is unbounded. Show that the sequence $(b_n)_{n\geq 1}$, $b_n = \{y_n\}$, diverges, where $\{x\}$ denotes the fractional part of the real number x.

177. Show that if the series $\sum_{n=1}^{\infty} \frac{1}{p_n}$ is convergent, where $p_1, p_2, \ldots, p_n$ are positive real numbers, then the series

$$\sum_{n=1}^{\infty} \frac{n^2}{(p_1 + p_2 + \cdots + p_n)^2} \cdot p_n$$

is also convergent.

178. (Carleman's inequality) Let $a_1, a_2, \ldots, a_n$ be a sequence of nonnegative real numbers. Prove that

$$\sum_{n=1}^{\infty} \sqrt[n]{a_1 a_2 \cdots a_n} \leq e \sum_{n=1}^{\infty} a_n.$$

179. (Hardy's inequality) Let $a_1, a_2, \ldots, a_n$ be a sequence of nonnegative real numbers and $p > 1$. Prove that

$$\sum_{n=1}^{\infty} \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p.$$

180. Let $f : [0, \infty) \to \mathbb{R}$ be a function satisfying $f(x)e^{f(x)} = x$ for $x \geq 0$. Prove that:
a) f is monotone;
b) $\lim_{x \to \infty} f(x) = \infty$;
c) $\frac{f(x)}{\log x}$ tends to 1 as $x \to \infty$.

181. Let $f : (0, \infty) \to \mathbb{R}$ be a function satisfying

$$\lim_{x \to \infty} \frac{f(x)}{x^k} = a, \qquad k \neq 1,$$

and

$$\lim_{x \to \infty} \frac{f(x+1) - f(x)}{x^{k-1}} = b \in \mathbb{R}.$$

Prove that $b = ka$.

182. Let $f : (a, b) \to \mathbb{R}$ be a function such that $\lim_{x \to x_0} f(x)$ exists for any $x_0 \in [a, b]$. Prove that f is bounded if and only if for all $x_0 \in [a, b]$, $\lim_{x \to x_0} f(x)$ is finite.

183. Prove that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \in \mathbb{Q} \implies f(x+1) \in \mathbb{R} \setminus \mathbb{Q}$.

184. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$f(x) \leq f\left(x + \frac{1}{n}\right),$$

for every real x and positive integer n. Show that f is nondecreasing.

185. Find all functions $f : (0, \infty) \to (0, \infty)$ subject to the conditions:
(i) $f(f(f(x))) + 2x = f(3x)$ for all $x > 0$;
(ii) $\lim\limits_{x\to\infty} (f(x) - x) = 0$.

186. Let $f(x)$ be a continuous function such that $f(2x^2 - 1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for $-1 \leq x \leq 1$.

187. Calculate

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right) \ln\left(1 + \frac{1}{2n}\right) \ln\left(1 + \frac{1}{2n+1}\right).$$

188. Let $f(x) = \sum\limits_{k=1}^{n} a_k \sin kx$, with $a_1, a_2, \ldots, a_n$ real numbers and $n \geq 1$. Prove that if $f(x) \leq |\sin x|$ for all $x \in \mathbb{R}$, then

$$\left| \sum_{k=1}^{n} k a_k \right| \leq 1.$$

189. Let $f : [a, b] \to \mathbb{R}$ be a function, continuous on $[a, b]$ and twice differentiable on (a, b). If $f(a) = f(b)$ and $f'(a) = f'(b)$, prove that for every real number λ, the equation

$$f''(x) - \lambda(f'(x))^2 = 0$$

has at least one zero in the interval (a, b).

190. Let $\alpha > 1$ be a real number, and let $(u_n)_{n\geq 1}$ be a sequence of positive numbers such that $\lim\limits_{n\to\infty} u_n = 0$ and $\lim\limits_{n\to\infty} \dfrac{u_n - u_{n+1}}{u_n^{\alpha}}$ exists and is nonzero. Prove that $\sum\limits_{n=1}^{\infty} u_n$ converges if and only if $\alpha < 2$.

191. Prove that, for any two bounded functions $g_1, g_2 : \mathbb{R} \to [1, \infty)$, there exist functions $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ such that for every $x \in \mathbb{R}$,

$$\sup_{s\in\mathbb{R}} \left(g_1(s)^x g_2(s)\right) = \max_{t\in\mathbb{R}} \left(xh_1(t) + h_2(t)\right).$$

192. Is there a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for all reals x?

193. Does there exist a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x) > 0$ and $f'(x) = f(f(x))$ for all reals x?

194. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

195. Find the set of all real numbers k with the following property: for any positive, differentiable function f that satisfies $f'(x) > f(x)$ for all x, there is some number N such that $f(x) > e^{kx}$ for all $x > N$.

196. Let $f : (1, \infty) \to \mathbb{R}$ be a differentiable function such that

$$f'(x) = \frac{x^2 - (f(x))^2}{x^2\left((f(x))^2 + 1\right)} \qquad \text{for all } x > 1.$$

Prove that $\lim_{x\to\infty} f(x) = \infty$.

197. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \le e \\ xf(\ln x) & \text{if } x > e \end{cases}$$

Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converge?

198. Find all sequences $a_0, a_1, \ldots, a_n$ of real numbers with $a_n \neq 0$, for which the following statement is true: if $f : \mathbb{R} \to \mathbb{R}$ is an n times differentiable function and $x_0 < x_1 < \cdots < x_n$ are real numbers such that $f(x_0) = f(x_1) = \cdots = f(x_n) = 0$, then there is $h \in (x_0, x_n)$ for which

$$a_0 f(h) + a_1 f'(h) + \cdots + a_n f^{(n)}(h) = 0.$$

199. Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function. Assume that

$$\lim_{x\to\infty} \left(f(x) + \frac{f'(x)}{x} \right) = 0.$$

Prove that $\lim_{x\to\infty} f(x) = 0$.

200. Let f be a twice continuously differentiable function on $(0, \infty)$ such that $\lim_{x\to 0^+} f'(x) = -\infty$ and $\lim_{x\to 0^+} f''(x) = +\infty$. Show that

$$\lim_{x\to 0^+} \frac{f(x)}{f'(x)} = 0.$$

201. Determine all Riemann integrable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\int_0^{x+1/n} f(t)\,dt = \int_0^x f(t)\,dt + \frac{1}{n} f(x),$$

for all reals x and all positive integers n.

202. For each continuous function $f : [0, 1] \to \mathbb{R}$, let $I(f) = \int_0^1 x^2 f(x)\,dx$ and $J(f) = \int_0^1 x f^2(x)\,dx$. Find the maximum value of $I(f) - J(f)$ over all such functions.

203. Let $f : [0, 1] \to [0, \infty)$ be a continuous function. Show that

$$\int_0^1 f^3(x)\,dx \geq 4 \int_0^1 x^2 f(x)\,dx \int_0^1 x f^2(x)\,dx.$$

204. Find all differentiable functions $f : [0, 1] \to \mathbb{R}$ with continuous derivative such that $f(0) = -\frac{1}{6}$ and

$$\int_0^1 (f'(x))^2\,dx \leq 2 \int_0^1 f(x)\,dx.$$

205. a) Let $f : [0,1] \to \mathbb{R}$ be continuously differentiable and satisfy $f(1) = 0$. Show that

$$\int_0^1 |f(x)|^2\,dx \le 4\int_0^1 x^2(f'(x))^2\,dx.$$

b) Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with continuous derivative such that $f(1) = 0$. Show that

$$4\int_0^1 x^2|f'(x)|^2\,dx \ge \int_0^1 |f(x)|^2\,dx + \left(\int_0^1 f(x)\,dx\right)^2.$$

206. Let $f : [0,1] \to [0,1)$ be a continuous function such that

$$\int_0^1 f(x^n)\,dx \le \int_0^1 f^n(x)\,dx, \qquad n \ge 1.$$

Show that $f(x) = 0$ for all $x \in [0,1]$.

207. Find all continuous functions $f : \mathbb{R} \to [1,\infty)$ for which there exists $a \in \mathbb{R}$ and a positive integer k such that

$$f(x)f(2x)\cdots f(nx) \le an^k,$$

for every real number x and positive integer n.

208. Let $f : [0,1] \to \mathbb{R}^+$ be an integrable function. Compute the following limits:

a) $\displaystyle\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{1}{n}f\left(\frac{k}{n}\right)\right)$;

b) $\displaystyle\lim_{n\to\infty}\left(\sum_{k=1}^{n}\exp\left(\frac{1}{n}f\left(\frac{k}{n}\right)\right)-n\right)$.

209. Prove that

$$\lim_{n\to\infty} n\left(\frac{\pi}{4} - n\int_0^1 \frac{x^n}{1+x^{2n}}\,dx\right) = \int_0^1 f(x)\,dx,$$

where $f(x) = \dfrac{\arctan x}{x}$ if $x \in (0,1]$ and $f(0) = 1$.

210. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $\displaystyle\int_0^1 f(x)\,dx = 0$. Prove that there is $c \in (0, 1)$ such that

$$\int_0^c xf(x)\,dx = 0.$$

211. Let $f : [-1, 1] \to \mathbb{R}$ be a continuous function. Show that:
a) if $\displaystyle\int_0^1 f(\sin(x + \alpha))\,dx = 0$ for every $\alpha \in \mathbb{R}$, then $f(x) = 0$ for all $x \in [-1, 1]$;
b) if $\displaystyle\int_0^1 f(\sin(nx))\,dx = 0$ for every $n \in \mathbb{Z}$, then $f(x) = 0$ for all $x \in [-1, 1]$.

212. Let $f : [0, \infty) \to \mathbb{R}$ be a periodic function, with period 1, integrable on $[0, 1]$. For a strictly increasing and unbounded sequence $(x_n)_{n\geq 0}$, $x_0 = 0$, with $\lim\limits_{n\to\infty} (x_{n+1} - x_n) = 0$, denote $r(n) = \max\{k \mid x_k \leq n\}$.
a) Show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{r(n)} (x_k - x_{k+1}) f(x_k) = \int_0^1 f(x)\,dx.$$

b) Show that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{r(n)} \frac{f(\log k)}{k} = \int_0^1 f(x)\,dx.$$

213. Let f and g be two continuous, distinct functions from $[0, 1]$ to $(0, +\infty)$ such that $\displaystyle\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx$. Let $(y_n)_{n\geq 1}$ be the sequence defined by

$$y_n = \int_0^1 \frac{f^{n+1}(x)}{g^n(x)}\,dx.$$

Prove that (y_n) is an increasing and divergent sequence.

214. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function such that

$$x \int_x^{x+1} f(t)\,dt = \int_0^x f(t)\,dt, \qquad \text{for any } x \in \mathbb{R}.$$

Prove that f is a constant function.

215. Let $f : [0,1] \to [0,\infty)$ be an integrable function. Show that

$$\int_0^1 f(x)\,dx \int_0^1 f^2(x)\,dx \leq \int_0^1 f^3(x)\,dx,$$

and

$$\int_0^1 xf(x)\,dx \int_0^1 x^2 f(x)\,dx \leq \int_0^1 f(x)\,dx \int_0^1 x^3 f(x)\,dx.$$

216. Show that for any continuous function $f : [0,1] \to \mathbb{R}$,

$$\frac{1}{4}\int_0^1 f^2(x)\,dx + 2\left(\int_0^1 f(x)\,dx\right)^2 \geq 3\int_0^1 f(x)\,dx \int_0^1 xf(x)\,dx.$$

217. Let $f : [0,1] \to \mathbb{R}$ be an integrable function such that

$$\int_0^1 f(x)\,dx = \int_0^1 xf(x)\,dx = \int_0^1 x^2 f(x)\,dx = 1.$$

Show that $\displaystyle\int_0^1 f^2(x)\,dx \geq 9$.

218. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with continuous derivative such that

$$\int_0^1 f(x)\,dx = \int_0^1 xf(x)\,dx = 1.$$

Show that $\displaystyle\int_0^1 (f'(x))^2\,dx \geq 30$.

219. a) Consider $f : [0,\infty) \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$ two continuous functions such that $\displaystyle\lim_{x\to\infty} f(x) = L \in \mathbb{R}$. Show that

$$\lim_{n\to\infty} \frac{1}{n}\int_0^n f(x)g\left(\frac{x}{n}\right)dx = L\int_0^1 g(x)\,dx.$$

b) Find $\displaystyle\lim_{n\to\infty} \frac{1}{n}\int_0^n \frac{x\log(1+x/n)}{1+x}\,dx$.

220. a) Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $\displaystyle\int_0^1 f(x)\,dx = 0$. Show that there exists $c \in (0,1)$ such that

$$f(c) = \int_0^c f(x)\,dx.$$

b) Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(1) = 0$. Show that there exists $c \in (0,1)$ such that

$$f(c) = \int_0^c f(x)\,dx.$$

221. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function for which there exist $a, b \in (0,1)$, $a < b$, such that $\displaystyle\int_0^a f(x)\,dx = 0$ and $\displaystyle\int_b^1 f(x)\,dx = 0$. Let $M = \displaystyle\sup_{x\in[0,1]} |f'(x)|$. Show that

$$\left|\int_0^1 f(x)\,dx\right| \le \frac{1-a+b}{4}M.$$

Moreover, if $f : [0,1] \to [0,\infty)$ is differentiable on $[0,1]$ and there exists $a \in (0,1]$ with $\displaystyle\int_0^a f(x)\,dx = 0$, show that

$$\left|\int_0^1 f(x)\,dx\right| \le \frac{1-a}{2} \sup_{x\in(0,1)} |f'(x)|.$$

222. a) Let $f_0 : [0,1] \to \mathbb{R}$ be a continuous function. Define the sequence of functions $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = \displaystyle\int_0^x f_{n-1}(t)\,dt$ for all integers $n \ge 1$. Prove that the series $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ is convergent for every $x \in [0,1]$ and find an explicit formula for the sum of the series.

b) Let $F_0 = \ln x$. For $n \ge 0$ and $x > 0$, let $F_{n+1}(x) = \displaystyle\int_0^x F_n(t)\,dt$. Evaluate $\displaystyle\lim_{n\to\infty} \frac{n!F_n(1)}{\ln n}$.

223. Let $f : [0,\infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\displaystyle\lim_{x\to\infty} f(x) = 0$. Prove that $\displaystyle\int_0^\infty \frac{f(x) - f(x+1)}{f(x)}\,dx$ diverges.

224. Let $(a_n)_{n\geq 1}$ be defined by

$$a_n = \frac{3}{2}\log n - \int_0^1 \log\left(1^t + 2^t + \cdots + n^t\right) dt.$$

i) Show that the sequence a_n is convergent and find its limit a.
ii) Show that $0 < n(a - a_n) < \dfrac{3}{2}$.

225. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and periodic function of period $T > 0$. Show that:
i) $\lim\limits_{x\to\infty} \dfrac{1}{x}\displaystyle\int_0^x f(t)\,dt = \dfrac{1}{T}\displaystyle\int_0^T f(t)\,dt.$
ii) $\lim\limits_{n\to\infty} \displaystyle\int_a^b f(nx)\,dx = \dfrac{b-a}{T}\displaystyle\int_a^b f(t)\,dt.$
iii) Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f(x+1) = f(x)$ and $g(x+1) = g(x)$ for all real x. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x)g(nx)\,dx = \int_0^1 f(x)\,dx \int_0^1 g(x)\,dx.$$

226. Let $a_1, a_2, \ldots$ be real numbers. Suppose there is a constant A such that for all n,

$$\int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} \frac{1}{1+(x-a_i)^2}\right)^2 dx \leq An.$$

Prove there is a constant $B > 0$ such that for all n,

$$\sum_{i,j=1}^{n} \left(1 + (a_i - a_j)^2\right) \geq Bn^3.$$

227. a) (Hermite–Hadamard) Let $f : [a, b] \to \mathbb{R}$ be a convex function. Show that

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)\,dx \leq (b-a)\frac{f(a)+f(b)}{2}.$$

b) Show that $\displaystyle\int_k^{k+1} \log x\,dx \geq \frac{\log k + \log(k+1)}{2}$ for $k \geq 1$, and $\displaystyle\int_{k-1/2}^{k+1/2} \log x\,dx \leq \log k$ for $k \geq 1$.
c) Consider the sequence $(a_n)_{n\geq 1}$ defined by

$$a_n = \int_1^n \log x\,dx - \log 2 - \cdots - \log(n-1) - \frac{1}{2}\log n, \qquad n \geq 1.$$

Show that a_n is increasing and $0 \le a_n \le \dfrac{1}{2}\log\dfrac{5}{4}$.

d) Prove that $\sqrt[4]{\dfrac{4}{5}}\, e\sqrt{n}\left(\dfrac{n}{e}\right)^n \le n! \le e\sqrt{n}\left(\dfrac{n}{e}\right)^n$ for $n \ge 1$.

e) (Stirling) Show that $\displaystyle\lim_{n\to\infty} \frac{n!}{\left(\frac{n}{e}\right)^n\sqrt{2\pi n}} = 1$.

228. Show that for any continuous function $f : [-1,1] \to \mathbb{R}$ we have the inequalities

$$\int_{-1}^{1} f^2(x)\,dx \ge \frac{1}{2}\left(\int_{-1}^{1} f(x)\,dx\right)^2 + \frac{3}{2}\left(\int_{-1}^{1} xf(x)\,dx\right)^2,$$

and

$$\int_{-1}^{1} f^2(x)\,dx \ge \frac{5}{2}\left(\int_{-1}^{1} x^2 f(x)\,dx\right)^2 + \frac{3}{2}\left(\int_{-1}^{1} xf(x)\,dx\right)^2.$$

229. a) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that $\displaystyle\lim_{n\to\infty}\int_0^1 x^n f(x)\,dx = 0$.

b) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n\to\infty} n\int_0^1 x^n f(x)\,dx = f(1), \qquad \lim_{n\to\infty} n\int_0^1 x^n f(x^n)\,dx = \int_0^1 f(x)\,dx.$$

c) Let $f, g : [0,1] \to \mathbb{R}$ be two continuous functions. Show that

$$\lim_{n\to\infty} \frac{\int_0^1 x^n f(x)\,dx}{\int_0^1 x^n g(x)\,dx} = \frac{f(1)}{g(1)}, \qquad \lim_{n\to\infty} \frac{\int_0^1 x^n f(x^n)\,dx}{\int_0^1 x^n g(x^n)\,dx} = \frac{\int_0^1 f(x)\,dx}{\int_0^1 g(x)\,dx}.$$

d) Find a real number c and a positive number L for which

$$\lim_{r\to\infty} r^c \frac{\int_0^{\pi/2} x^r \sin x\,dx}{\int_0^{\pi/2} x^r \cos x\,dx} = L.$$

230. Let $(a_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$a_0 = 1, \qquad a_{n+1} = \frac{1}{n+1}\sum_{k=0}^{n} \frac{a_k}{n-k+2}.$$

Find the limit $\displaystyle\lim_{n\to\infty}\sum_{k=0}^{n}\frac{a_k}{2^k}$.

231. Prove that for any real numbers $a_1, a_2, \ldots, a_n$,

$$\sum_{1 \le i,j \le n} \frac{ij}{i+j-1} a_i a_j \ge \left(\sum_{i=1}^{n} a_i \right)^2 .$$

232. (Hardy) Let $p > 1$ and let $f : [0, \infty) \to \mathbb{R}$ be a differentiable increasing function such that $f(0) = 0$ and $\displaystyle\int_0^\infty (f'(x))^p \, dx$ is finite. Show that

$$\int_0^\infty x^{-p} |f(x)|^p \, dx \le \left(\frac{p}{p-1} \right)^p \int_0^\infty (f'(x))^p \, dx.$$

233. Let $u_1, u_2, \ldots, u_n \in C([0,1]^n)$ be nonnegative and continuous functions, where u_j does not depend on the j-th variable for $j = 1, 2, \ldots, n$. Show that

$$\left(\int_{[0,1]^n} \prod_{j=1}^{n} u_j \right)^{n-1} \le \prod_{j=1}^{n} \int_{[0,1]^n} u_j^{n-1}.$$

234. (Zagier) Prove that if $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are nonnegative, then

$$\left(\sum_{1 \le i,j \le n} \min(a_i, a_j) \right) \left(\sum_{1 \le i,j \le n} \min(b_i, b_j) \right) \ge \left(\sum_{1 \le i,j \le n} \min(a_i, b_j) \right)^2 .$$

235. Let $\{D_1, D_2, \ldots, D_n\}$ be a set of disks in the Euclidean plane and $a_{ij} = S(D_i \cap D_j)$ be the area of $D_i \cap D_j$. Prove that for any real numbers $x_1, x_2, \ldots, x_n$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \ge 0.$$

236. Let $a, b \in \mathbb{R}$, $f : [a, b] \to [0, 1]$ an increasing function, and define the sequence $(c_n)_{n \ge 1}$ by $c_n = n \displaystyle\int_a^b f^n(x) \, dx$. Show that:

i) $\displaystyle\lim_{n \to \infty} (c_{n+1} - c_n) = 0.$

ii) $\displaystyle\lim_{n \to \infty} \frac{c_{n+1}}{c_n} = f(b^-).$

237. Let $f : [0,1] \to \mathbb{R}$ be a C^1 function. Show that

$$\lim_{n\to\infty} n \left[\frac{1}{2}\left(\int_0^1 f(x)\,dx\right)^2 - \frac{1}{n^2}\sum_{1\le i,j\le n} f\left(\frac{i}{n}\right) f\left(\frac{j}{n}\right)\right] = \frac{1}{2}\int_0^1 f^2(x)\,dx + \frac{f(0)-f(1)}{2}\int_0^1 f(x)\,dx.$$

238. a) Find the limit $\displaystyle\lim_{n\to\infty}\left(\int_0^1 (1+x^n)^n\,dx\right)^{1/n}$.

b) Prove that $\displaystyle\lim_{n\to\infty} n^2\left(\int_0^1 \sqrt[n]{1+x^n}\,dx - 1\right) = \frac{\pi^2}{12}$.

239. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function such that f' is bounded. Show that

$$\int_0^1 f^2(x)\,dx - \left(\int_0^1 f(x)\,dx\right)^2 \le \frac{1}{12}\left(\sup_{x\in[0,1]} |f'(x)|\right)^2.$$

240. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous injection. Prove that if there exists n such that the n-th iterate of f is the identity, that is, $f^n(x) = x$ for all $x \in \mathbb{R}$, then:
a) $f(x) = x$ for all $x \in \mathbb{R}$, if f is strictly increasing;
b) $f^2(x) = x$ for all $x \in \mathbb{R}$, if f is strictly decreasing.

241. (Continuity of square-root-type functional equations)
a) Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x)^2 = -x$ for all $x \in \mathbb{R}$. Show that f cannot be continuous.
b) Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f^2(x) = \cos(x)$ for all $x \in \mathbb{R}$.

242. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which have the Intermediate Value Property and such that there is $n \in \mathbb{N}$ for which $f^n(x) = -x$ for all $x \in \mathbb{R}$.

243. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ which attains each of its values exactly three times. Does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ which attains each of its values exactly two times?

244. Let $f \in C^2[0,1]$ be such that $f(0) = f(1)$. Show that

$$3f'(1)^2 \le \int_0^1 f''(x)^2\,dx.$$

245. (Monotonic functions and continuity)
a) If f is monotonic on $[a, b]$, then the set of discontinuities of f is countable.
b) Let $f : \mathbb{R} \to \mathbb{R}$ be a surjective monotonous function. Show that f must be continuous.

246. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Consider any sequence given by $x_{n+1} = f(x_n)$. Show that (x_n) converges if and only if $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$.

247. Let f be a three times continuously differentiable function with the property that $f(0) = f'(0) = 0 < f''(0)$. Define

$$g(x) = \frac{d}{dx}\left(\frac{\sqrt{f(x)}}{f'(x)}\right)$$

for $x \neq 0$ and $g(0) = 0$. Show that g is bounded in a neighbourhood of 0. Does this conclusion still follow if f is only twice continuously differentiable?

248. Let $f : [0, 1] \to \mathbb{R}$ be given by $f(x) = 2x(1 - x)$. Denote $f_n = f \circ \cdots \circ f$ for the n-fold composition of f. Determine

$$\int_0^1 f_n(x)\, dx.$$

Does this sequence converge?

249. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function satisfying $f(0) = 2$, $f'(0) = -2$ and $f(1) = 1$. Prove that there exists some $c \in (0, 1)$ such that

$$f(c) \cdot f'(c) + f''(c) = 0.$$

250. Let $c \in (0, 1)$ and define

$$f(x) = \begin{cases} x/c & \text{if } x \in [0, c]; \\ (1 - x)/(1 - c) & \text{if } x \in [c, 1]. \end{cases}$$

Show that the equation $f^n(x) = x$ only has finitely many solutions for every $n \geq 1$, where f^n denotes the n-fold composition of f with itself. Show also that for every n, there are solutions to $f^n(x) = x$ for which this n is minimal.

251. (*) Let $f : (0,1) \to [0,\infty)$ be a function that is non-zero only on the distinct points $a_1, a_2, \ldots$. Suppose that

$$\sum_{n=1}^{\infty} f(a_n) < \infty.$$

Prove that f is differentiable at at least one $x \in (0,1)$.

252. Let $f : [0,1] \to [0,1]$ be a strictly increasing function. Show that for some $x \in [0,1]$, it holds that $f(x) = x$. What if f is strictly decreasing?

253. Let $f : [a,b] \to [a,b]$ be a continuous function and let $p \in [a,b]$. Define the sequence (p_n) by $p_0 = p$ and $p_{n+1} = f(p_n)$ for all $n \geq 0$. Suppose that the set $T_p = \{p_n \mid n \in \mathbb{N}\}$ is closed. Prove that T_p is finite.

254. (*) Let $g : [0,1] \to \mathbb{R}$ be a function and define the sequence of functions $f_n : (0,1] \to \mathbb{R}$ by setting $f_0(x) = g(x)$ and

$$f_{n+1}(x) = \frac{1}{x}\int_0^x f_n(t)\,dt$$

for all $x \in (0,1]$ and $n \geq 0$. Determine $\lim_{n\to\infty} f_n(x)$ for every $x \in (0,1]$.

255. Let $f, g : [a,b] \to [0,\infty)$ be continuous and non-decreasing functions such that for each $x \in [a,b]$, we have

$$\int_a^x \sqrt{f(t)}\,dt \leq \int_a^x \sqrt{g(t)}\,dt,$$

with equality for $x = b$. Prove that

$$\int_a^b \sqrt{1+f(t)}\,dt \geq \int_a^b \sqrt{1+g(t)}\,dt.$$

256. Let $f : \mathbb{R} \to \mathbb{R}$ be a three times differentiable function. Prove that for some $c \in (-1,1)$, it holds that

$$f'''(c) = 3f(1) - 3f(-1) - 6f'(0).$$

257. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions with the property that $f(r) \leq g(r)$ for all $r \in \mathbb{Q}$. If f and g are non-decreasing, does this imply that $f(x) \leq g(x)$ for all $x \in \mathbb{R}$? What if f and g are both continuous?

258. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. We say some $x \in \mathbb{R}$ is a shadow-point of f if for some $y > x$ it holds that $f(y) > f(x)$. Let $a < b$ now be two real numbers that are not shadow-points of f, with the property that all points in (a, b) are shadow-points of f. Show that $f(x) \leq f(b)$ for all $x \in (a, b)$ and that $f(a) = f(b)$.

259. Compute

$$\lim_{A\to\infty} \frac{1}{A} \int_1^A A^{1/x}\, dx.$$

260. Let $f : [a, b] \to \mathbb{R}$ be a continuous function that is differentiable on (a, b). Suppose that f has infinitely many zeroes, but that there is no $x \in (a, b)$ such that $f(x) = f'(x) = 0$. Prove that $f(a)f(b) = 0$ and give an example of such a function on $[0, 1]$.

261. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function satisfying the property that $\lim_{x\to\infty} f(x) = L$ exists. Prove that

$$\lim_{n\to\infty} \int_0^1 f(nx)\, dx = L.$$

262. (*) Define the sequence $f_1, f_2, \ldots : [0, 1] \to \mathbb{R}$ of continuously differentiable functions by

$$f_1 = 1, \qquad f'_{n+1} = f_n f_{n+1} \text{ on } (0, 1) \qquad \text{and} \qquad f_{n+1}(0) = 1.$$

Show that $\lim_{n\to\infty} f_n(x)$ exists for every $x \in [0, 1]$ and determine the limit function.

263. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that g is differentiable. Suppose that

$$(f(0) - g'(0))(g'(1) - f(1)) > 0.$$

Show that there exists some $c \in (0, 1)$ such that $f(c) = g'(c)$.

264. Let V be the set of all continuous functions $f : [0, 1] \to \mathbb{R}$, differentiable on $(0, 1)$, with the property that $f(0) = 0$ and $f(1) = 1$. Determine all $\alpha \in \mathbb{R}$ such that for every $f \in V$, there exists some $\xi \in (0, 1)$ such that $f(\xi) + \alpha = f'(\xi)$.

265. Determine all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ satisfying for all $x \in \mathbb{R}$ the equation

$$f(x)^2 = \int_0^x f(t)^2 + f'(t)^2\, dt + 2023.$$

266. Consider a continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(2x - f(x)) = x \quad \text{for all } x \in \mathbb{R}.$$

Suppose that $f(\alpha) = \alpha$ for some $\alpha \in \mathbb{R}$. Show that then $f(x) = x$ for all $x \in \mathbb{R}$. Does this conclusion still hold if f such an α need not necessarily exist?

267. Let $c > 0$. Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$f(x) = f(x^2 + c) \quad \text{for all } x \in \mathbb{R}.$$

268. Determine all $\alpha > 0$ with the property that for all differentiable functions $f : \mathbb{R} \to \mathbb{R}_{>0}$ such that $f'(x) > f(x)$ for all $x \in \mathbb{R}$, there exists some $N \in \mathbb{R}$ such that $f(x) > e^{\alpha x}$ for all $x > N$.

269. Let $a, b \in (0, 1/2)$ and let g be a continuous function satisfying the property that

$$g(g(x)) = ag(x) + bx \quad \text{for all } x \in \mathbb{R}.$$

Prove that $g(x) = cx$ for some $c \in \mathbb{R}$.

270. Prove that there exists no function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(f(x) + y) = f(x) + 3x + yf(y) \quad \text{for all } x, y \in \mathbb{R}.$$

271. Determine all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there exists some $a \in \mathbb{R}$ such that

$$f'(a/x) = x/f(x) \quad \text{for all } x > 0.$$

272. Determine all functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ satisfying

$$xf(x^2 f(y)) = yf(x) \quad \text{for all } x, y \in \mathbb{R}.$$

273. Determine all twice differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with $f'(x)f''(x) = 0$ for all $x \in \mathbb{R}$.

274. Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x) + \int_0^x (x - t)f(t)\, dt = 1 \quad \text{for all } x \in \mathbb{R}.$$

275. Let $f : [-2, 2] \to \mathbb{R}$ be continuously differentiable. Prove that there exists some $x \in (-2, 2)$ such that

$$f'(x) - f(x)^2 < 1.$$

276. Let $f : (0, \infty) \to \mathbb{R}$ be a twice continuously differentiable function such that

$$|f''(x) + 2xf'(x) + (x^2 + 1)f(x)| \leq 1$$

for all $x > 0$. Prove that $\lim_{x\to\infty} f(x) = 0$.

277. For any given $x \in (0, \pi/2)$ determine which of $\tan(\sin(x))$ and $\sin(\tan(x))$ is bigger.

278. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function satisfying $f(0) = 1$ and $f'(0) = 0$, and further

$$f''(x) - 5f'(x) + 6f(x) \geq 0 \quad \text{for all } x \geq 0.$$

Show that for all $x \geq 0$, it holds that

$$f(x) \geq 3e^{2x} - 2e^{3x}.$$

279. Let $0 < a < b$. Prove that

$$\int_a^b (x^2 + 1)e^{-x^2}\, dx \geq e^{-a^2} - e^{-b^2}.$$

280. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with the property that

$$f'(t) > f(f(t)) \quad \text{for all } t \in \mathbb{R}.$$

Show that

$$f(f(f(t))) \leq 0 \quad \text{for all } t \geq 0.$$

281. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function with the property that $f(0) = 0$. Show that there exists some $c \in (-\pi/2, \pi/2)$ such that

$$f''(c) = f(c)(1 + 2\tan(c)^2).$$

282. (*) Define $f(x) = \sin(x)/x$. Show that for all $x > 0$ and $n \geq 1$, it holds that

$$|f^{(n)}(x)| < \frac{1}{n+1},$$

where $f^{(n)}$ denotes the nth derivative of f.

283. Consider the set of continuous functions $f : [0, 1] \to \mathbb{R}$ satisfying

$$f(x) + f(y) \geq |x - y| \quad \text{for all } x, y \in [0, 1].$$

Find the minimal value of

$$\int_0^1 f(x)\, dx$$

for functions in this set.

284. Consider a differentiable function $f : \mathbb{R} \to (0, \infty)$ with the property that for some constant $L > 0$, it holds that

$$|f'(x) - f'(y)| \leq L|x - y| \quad \text{for all } x, y \in \mathbb{R}.$$

Show that for all $x \in \mathbb{R}$, it holds that

$$f'(x)^2 < 2Lf(x).$$

285. Find all differentiable functions $f : (0, \infty) \to \mathbb{R}$ with the property that

$$f(b) - f(a) = (b - a)f'(\sqrt{ab}) \quad \text{for all } a, b > 0.$$

286. Find all twice continuously differentiable functions $f : \mathbb{R} \to (0, \infty)$ such that

$$f''(x)f(x) \geq 2f'(x)^2 \quad \text{for all } x \in \mathbb{R}.$$

287. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that have a continuous second derivative and for which the equality

$$f(7x + 1) = 49f(x)$$

holds for all $x \in \mathbb{R}$.

288. If f has continuous derivative on $[-1, 1]$, find

$$\lim_{h \to 0^+} \int_{-1}^1 \frac{h\, f(x)}{h^2 + x^2}\, dx,$$

assuming the limit exists.

289. Let $f : [0, 1] \to \mathbb{R}$ be twice differentiable on $[0, 1]$ with $f''(x) > 0$ on $[0, 1]$. Show that

$$4\int_{1/2}^1 f(x)\, dx \leq f(1) + \int_0^1 f(x)\, dx.$$

290. If f has continuous third derivative on $[a, b]$ and has at least three roots in $[a, b]$, show that

$$\max_{x\in[a,b]} |f(x)| \leq \frac{(b-a)^3}{6} \max_{x\in[a,b]} |f^{(3)}(x)|.$$

291. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences defined by $a_0 > 0$, $b_0 > 0$,

$$a_{n+1} = a_n + \frac{1}{2b_n}, \qquad b_{n+1} = b_n + \frac{1}{2a_n}, \qquad n = 0, 1, 2, \ldots$$

Prove that $\max(a_{2019}, b_{2019}) > \dfrac{89}{2}$.

292. Let $u_1, u_2, \ldots$ be defined by $u_1 = 1$, $u_n = u_{n-1} + \dfrac{1}{u_{n-1}}$, $n = 2, 3, \ldots$. Prove that $63 < u_{2019} < 78$.

293. Let $(x_n)_{n=1}^{\infty}$ be defined by $x_1 = 1$, $x_{n+1} = \dfrac{\sqrt{x_n^2 + 2019x_n} + x_n}{2}$ for $n \geq 1$.

a) Let $y_n = \displaystyle\sum_{i=1}^{n} \frac{1}{x_i^2}$. Compute $\lim_{n\to+\infty} y_n$.

b) Compute $\displaystyle\lim_{n\to+\infty} \frac{x_n}{n}$.

294. Let $(x_n)_{n=1}^{\infty}$ be defined by $x_1 = \dfrac{1}{2}$, $x_{n+1} = x_n^2 - \dfrac{1}{2}$ for $n \geq 1$.

a) Prove that $-\dfrac{1}{2} < x_n < 0$ for all $n \geq 2$.

b) Prove that $(x_n)_{n=1}^{\infty}$ converges and find its limit.

295. Let $\{u_n\}$ be defined by $u_0 = 0$, $u_1 = \dfrac{1}{2}$, $u_{n+1} = \dfrac{1}{3}\left(1 + u_n + u_{n-1}^3\right)$ for $n = 1, 2, \ldots$. Prove that $\{u_n\}$ has a limit and find $\displaystyle\lim_{n\to\infty} u_n$.

296. Let $f : (-\infty, +\infty) \to (-\infty, +\infty)$ be continuous with $f(f(x)) > x$ for all $x \in \mathbb{R}$. Define $x_1 = f(0)$, $x_{n+1} = f(x_n)$ for $n \in \mathbb{N}^*$.

a) Prove that $x_n > 0$ for all $n \in \mathbb{N}^*$.

b) Find $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{x_{k+2} - x_k}{x_k x_{k+1} x_{k+2}}$.

297. Let $f : (-\infty, +\infty) \to (-\infty, +\infty)$ be bounded below. Prove that there exists a real sequence $\{x_n\}_{n=1}^{\infty}$ such that

$$f(x_n + 2019) - f(x_n) < \frac{1}{n} \qquad (\forall n \in \mathbb{N}^*).$$

298. Let f be twice differentiable on $\mathbb{R}$ with $\sup\limits_{x\in\mathbb{R}} |f(x)| < \infty$ and $\sup\limits_{x\in\mathbb{R}} |f''(x)| < \infty$. Prove that

$$\sup_{x\in\mathbb{R}} |f'(x)| \le 2\sqrt{\sup_{x\in\mathbb{R}} |f(x)| \cdot \sup_{x\in\mathbb{R}} |f''(x)|}.$$

299. Let $f : [a, b] \subset (0, +\infty) \to \mathbb{R}$ be continuously differentiable. Prove that there exist $c, d \in (a, b)$ with $c < d$ such that

$$c(c-a)f'(c) = a(b-c)f'(d).$$

300.

a) Let $f(x)$ be continuous on $[0, \frac{\pi}{2}]$ and differentiable on $(0, \frac{\pi}{2})$, with $f(0) = f(\frac{\pi}{2}) = 0$ and $f^2(x) + [f'(x)]^2 \neq 0$ for all $x \in (0, \frac{\pi}{2})$. Prove that there exists $c \in (0, \frac{\pi}{2})$ such that

$$\tan c = \frac{f(c) + f'(c)}{f(c) - f'(c)}.$$

b) Let $f : [0, 1] \to [0, \infty)$ be continuous. Study the convergence of the sequence

$$u_n = n \int_0^1 [f(t)]^n \, dt, \qquad n = 1, 2, 3, \ldots$$

301. Let f be four times continuously differentiable on $[-1, 1]$ with $f'(-1) = f'(1)$. Prove that there exists $c \in (-1, 1)$ such that

$$2f(0) - f(-1) - f(1) = \frac{1}{12} f^{(4)}(c).$$

302. Let f have continuous derivative on $[0, 1]$ with $f(0) = f(1) = 0$. Prove that there exists $x_0 \in [0, 1]$ such that

$$|f'(x_0)| \ge 4 \int_0^1 |f(x)| \, dx.$$

303. Let $f : [0,1] \to \mathbb{R}$ be twice differentiable, with $f(1) \geq 0$ and $f''(x) \leq 1$ for all $x \in [0,1]$. Prove that

$$f(0) - 2f\left(\tfrac{1}{2}\right) \leq \frac{1}{4}.$$

304. Let f be continuous on $[a,b]$ and g differentiable on $[a,b]$, with $(f(a) - g'(a))(f(b) - g'(b)) < 0$. Prove that there exists $c \in (a,b)$ such that $f(c) = g'(c)$.

305. Find all functions $f : [0,1] \to [0,1]$ satisfying

$$|f(x) - f(y)| \geq |x - y|, \qquad \forall x, y \in [0,1].$$

306. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x) = -f(x^2)$ for all $x \in \mathbb{R}$.

307. Let $f(x) = 2x - [x]$, $x \in \mathbb{R}$, where $[x]$ denotes the integer part of x.

1. Study the continuity of $f(x)$ on $\mathbb{R}$.
2. Prove that $f(x)$ is Riemann integrable on $[0,3]$, and compute $\displaystyle\int_0^3 f(x)\,dx$.

308. Let $f : \mathbb{R} \to [0, +\infty)$ be continuously differentiable. Prove that

$$\left| \int_0^1 f^3(x)\,dx - f^2(0) \int_0^1 f(x)\,dx \right| \leq \max_{0 \leq x \leq 1} |f'(x)| \left(\int_0^1 f(x)\,dx \right)^2.$$

309. Let f be continuous on $[0,1]$, differentiable on $(0,1)$, and satisfy

$$f(1) = 3 \int_0^{1/3} e^{1-x^2} f(x)\,dx.$$

Prove that there exists $c \in (0,1)$ such that $f'(c) = 2cf(c)$.

310. Let $f : [0,1] \to \mathbb{R}$ be continuously differentiable on $[0,1]$, with $f(1) = 0$. Prove that there exists $c \in (0,1)$ such that

$$2019\, f'(c) \int_0^c f(x)\,dx = -f(c) \cdot \tan(2019 f(c)).$$

311. Let $f:[0,1]\to[0,+\infty)$ be continuously differentiable on $[0,1]$, with $f(0)=0$, $\max\limits_{[0,1]} f'(x)=6$, $\displaystyle\int_0^1 f(x)\,dx=\frac{1}{3}$.

a) Prove that $0\le f(1)\le 2$.

b) Prove that $\displaystyle\int_0^1 [f(x)]^3\,dx\le\frac{2}{3}$.

312. Let φ and f be defined on $\mathbb{R}$ by

$$\varphi(x)=\begin{cases} e^{-1/x^2} & x\neq 0,\\ 0 & x=0,\end{cases}\qquad f(x)=\begin{cases} e^{1/x^2}\displaystyle\int_0^x \varphi(t)\,dt & x\neq 0,\\ 0 & x=0.\end{cases}$$

It is known that φ is continuous and differentiable on $\mathbb{R}$, and f is odd on $\mathbb{R}$.

1. Prove that f is differentiable at every $x\neq 0$ and compute $f'(x)$.
2. Prove that for all $x\ge 0$,

$$\int_0^x \varphi(t)\,dt=\frac{x^3}{2}\varphi(x)-\frac{3}{2}\int_0^x t^2\varphi(t)\,dt.$$

   Deduce that f is differentiable at 0 and $f'(0)=0$.
3. Compute $\displaystyle\lim_{x\to 0}\frac{f(x)}{x^3}$.

313. Let $f:[0,1]\to\mathbb{R}$ be differentiable and satisfy $\displaystyle\int_0^1 f(x)\,dx=\int_0^1 xf(x)\,dx$. Prove that there exists $c\in(0,1)$ such that

$$f(c)=2018\,f'(c)\int_0^c f(x)\,dx.$$

314. Let $a<b$ be real numbers and $f:[a,b]\to\mathbb{R}$ continuously differentiable with $\displaystyle\int_a^b f(x)\,dx=0$. Prove that

$$\max_{x\in[a,b]}\left|\int_a^x f(t)\,dt\right|\le\frac{(b-a)^2}{8}\max_{x\in[a,b]}|f'(x)|.$$

315. Let $f(x)$ be differentiable on $[a,b]$ and satisfy $f(a)=f(b)=0$, $f(x)\neq 0$ for all $x\in(a,b)$. Prove that there exists a sequence $\{x_n\}$, $x_n\in(a,b)$, such that

$$\lim_{n\to\infty}\frac{f'(x_n)}{(\sqrt[n]{e}-1)f(x_n)}=2021.$$

316. Let $f(x)$ be continuous on $[0,1]$, differentiable on $(0,1)$, and satisfy $f(0)=f(1)=0$. Prove that there exists $c\in(0,1)$ such that $f'(c)=f(c)$.

317.

1. Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), with $0\le a<b$. Prove that there exist $c_1,c_2\in(a,b)$ such that

$$\frac{f(a)f'(c_2)+f(c_2)f'(c_1)}{b-c_2}=f'(c_1)f'(c_2).$$

2. Suppose $f:[0,1]\to(0,\infty)$ is twice differentiable, with $f(0)=f'(0)=2$ and $f(1)=1$. Prove that there exists $\xi\in(0,1)$ such that

$$f(\xi)f'(\xi)+f''(\xi)=0.$$

318. Let $f:[0,1]\to\mathbb{R}$ be continuously differentiable with $\max\limits_{t\in[0,1]}|f'(t)|=M<\infty$. Prove that for all $x\in[0,1]$,

$$\left|f(x)-\int_0^1 f(t)\,dt\right|\le\frac{1}{4}\left[(2x-1)^2+1\right]M,$$

and that the constant $\frac{1}{4}$ is best possible (it cannot be replaced by a smaller number).

319. Let $f:[0;1]\to\mathbb{R}$ be continuous with $\displaystyle\int_0^1 f(x)\,dx=0$. Prove that

$$\int_0^1 f^2(x)\,dx\ge 12\left(\int_0^1 xf(x)\,dx\right)^2.$$

320. Let $f(x)$ be continuously differentiable on $[0,1]$. Prove that

$$\left|f\left(\tfrac{1}{2}\right)\right|\le\int_0^1\left(|f(t)|+\tfrac{1}{2}|f'(t)|\right)dt,$$

$$|f(x)|\le\int_0^1\left(|f(t)|+|f'(t)|\right)dt,\qquad\forall x\in[0,1].$$

321. Let $f : [-1, 1] \to \mathbb{R}$ be continuous. Prove that

$$\int_{-1}^{1} [f(x)]^2 \, dx \geq \frac{5}{2} \left(\int_{-1}^{1} x^2 f(x) \, dx \right)^2 + \frac{3}{2} \left(\int_{-1}^{1} x f(x) \, dx \right)^2 .$$

322. Let $f \in C^2[-1, 1]$ (i.e., f is twice continuously differentiable on $[-1, 1]$) with $f(0) = 0$. Prove that

$$\int_{-1}^{1} [f''(x)]^2 \, dx \geq 10 \left[\int_{-1}^{1} f(x) \, dx \right]^2 .$$

323. Let $f : [a, b] \to \mathbb{R}$ be twice differentiable with $f'(a) = f'(b) = 0$. Prove that there exists $c \in (a, b)$ such that

$$|f''(c)| \geq \frac{4}{(b-a)^2} |f(b) - f(a)|.$$

324. Let $f(x)$ be continuous on $[0, 1]$. Let $m = \min\limits_{x \in [0,1]} f(x)$, $M = \max\limits_{x \in [0,1]} f(x)$. Prove that

$$\int_{0}^{1} [f(x)]^2 \, dx - \left[\int_{0}^{1} f(x) \, dx \right]^2 \leq \frac{(M-m)^2}{4}.$$

325. Let $f : [1, 13] \to \mathbb{R}$ be a convex, integrable function. Prove that

$$\int_{1}^{3} f(x) \, dx + \int_{11}^{13} f(x) \, dx \geq \int_{5}^{9} f(x) \, dx.$$

326. Let f be defined on $\mathbb{R}$, periodic with period $T > 0$, and integrable on $[0, T]$. Compute

$$\lim_{x \to \infty} \frac{\int_0^x f(t) \, dt}{x}.$$

327. Let $f : [0, 1] \to \mathbb{R}$ be continuous with $\displaystyle\int_{0}^{1} [f(x)]^3 \, dx = 0$. Show that

$$\int_{0}^{1} [f(x)]^4 \, dx \geq \frac{27}{4} \left[\int_{0}^{1} f(x) \, dx \right]^4 .$$

328. Let $f : [a, b] \to \mathbb{R}$ be continuous with $\displaystyle\int_a^b f(x)\,dx = 0$. Prove that there exists $\alpha \in \left(0, \dfrac{b-a}{2}\right)$ such that

$$f\left(\frac{a+b}{2} - \alpha\right) + f\left(\frac{a+b}{2} + \alpha\right) = 0.$$

329. Let $\mathcal{M}$ be the set of continuous functions $f : [0; 1] \to \mathbb{R}$ satisfying:

a) $\displaystyle\int_a^b f(x)\,dx = \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$ for all real a, b with $0 \leq a < b \leq 1$;

b) $f(x)$ always has a root in $[0, 1]$, and the set of roots of $f(x)$ in $[0, 1]$ is finite.

Prove that:

1) Every cubic polynomial satisfies condition a).

2) $\displaystyle\int_0^1 |f(x)|\,dx \leq \frac{5}{6}\max_{[0,1]}|f(x)|$ for every $f \in \mathcal{M}$.

3) For every $C \in \left(0, \dfrac{5}{6}\right)$, there exists at least one $f \in \mathcal{M}$ such that

$$\int_0^1 |f(x)|\,dx > C\max_{[0,1]}|f(x)|.$$

330. Let f be continuously differentiable on $[a, b]$ with $\displaystyle\int_a^b f(x)\,dx = 0$. Let $M = \displaystyle\max_{x\in[a,b]}|f'(x)|$. Prove that

$$\left|\int_a^b xf(x)\,dx\right| \leq \frac{(b-a)^3}{12}M.$$

331. Let $f : (0, +\infty) \to \mathbb{R}$ have a continuous second derivative and satisfy

$$|f''(x) + 4xf'(x) + 2(2x^2 + 1)f(x)| \leq 2018, \qquad \forall x \geq 0.$$

Prove that $\displaystyle\lim_{x\to+\infty} f(x) = 0$.

332. Let f be continuously differentiable on $[2018, 2019]$ with $f(2019) = 0$. Prove that the equation

$$(x - 2018)f'(x) = (x - 2019)f(x)$$

has a solution.

333. Let $f : \mathbb{R} \to \mathbb{R}$ have a second derivative and satisfy $f\left(\frac{\pi}{2}\right) = 0$. Prove that there exists $c \in (0, \pi)$ such that

$$f(c) - f''(c) = 2f'(c)\cot(c).$$

334. Let $f(x)$ be continuously differentiable on $[0, 1]$ with $\displaystyle\int_0^1 \big(f'(x) + (1 - 2x)f(x)\big)\, dx = 5$. Prove that there exists $c \in (0, 1)$ such that $f'(c) = 6$.

335. Let $f : [0, 1] \to [-1, 1]$ be continuously differentiable on $[0, 1]$ with $f(0) = f(1) = 0$. Prove that there exists $c \in (0, 1)$ such that

$$f'(c) = 2c \tan f(c).$$

336. Let f be continuous on $[2013, 2015]$ with $\displaystyle\int_{2013}^{2015} f(x)\, dx = 0$. Prove that there exists $c \in (2013, 2015)$ such that

$$2014 \int_{2013}^{c} f(x)\, dx = cf(c).$$

337. Let $f : [0, 1] \to \mathbb{R}$ be continuous with $\displaystyle\int_0^1 f(x)\, dx = \int_0^1 xf(x)\, dx$. Prove that there exists $c \in (0, 1)$ such that

$$f(c) = 2014 \int_0^c f(x)\, dx.$$

338. Let f be continuous on $[a, b]$ and differentiable on (a, b), with $a > 0$ and $\displaystyle\int_a^b f(x)\, dx = 0$. Prove that there exists $c \in (a, b)$ such that

$$2014c \int_a^c f(x)\, dx - 2013cf(c) + 2012 \int_a^c f(x)\, dx = 0.$$

339. Let $f : [-1;1] \to \mathbb{R}$ be continuous. Prove that the equation

$$xf^{2014}(x) - 2014f(x) = -2013x$$

has a solution on $[-1,1]$.

340. Let $\varphi(x)$ have a first derivative on $[1,2]$ and a second derivative on $(1,2)$, satisfying

$$\int_1^2 \varphi(x)\,dx = 2\varphi(2) - \varphi(1) - 1, \qquad \varphi'(1) = 1.$$

Prove that the equation $x\varphi''(x) + \varphi'(x) = 0$ always has a solution in $(1,2)$.

341. Let f and g be both increasing or both decreasing on $[0,1]$. Prove that

$$\int_0^1 f(x)g(x)\,dx \geq \int_0^1 f(x)\,dx \cdot \int_0^1 g(x)\,dx.$$

342. Let $f(x) = \dfrac{x^2 - x}{x^3 - 3x + 1}$. Prove that

$$I = \int_{-100}^{-10} f^2(x)\,dx + \int_{1/101}^{1/11} f^2(x)\,dx + \int_{101/100}^{11/10} f^2(x)\,dx$$

is a rational number.

343. Let f be continuous on $[a,b]$ with $f''(x) \geq 0$ for all $x \in (a,b)$. Prove that

$$\int_a^b f(x)\,dx \geq (b-a)\left(f(c) + \left(\frac{a+b}{2} - c\right)f'(c)\right), \qquad \forall c \in (a,b).$$

344. Let f be defined and continuously differentiable on $[0,b]$, with f' decreasing on $[0,b]$. Prove that for all $a \in (0,b)$,

$$\frac{1}{b^2}\int_0^b f(x)\,dx - \frac{1}{a^2}\int_0^a f(x)\,dx \geq \frac{f(b)}{b} - \frac{f(a)}{a}.$$

345. Given the sequence of real numbers (a_n) satisfying

$$a_{n+1} = \frac{n+1}{n}a_n - \frac{2}{n}, \qquad a_1 = \alpha, \qquad n = 1, 2, 3, \ldots$$

Find all α for which (a_n) converges.

346. Let $s < q$ be real numbers, p a given natural number, and let $a_n^{(p)}, a_n^{(p-1)}, \ldots, a_n^{(0)}$ be sequences of real numbers such that

$$\lim_{n\to\infty} \int_s^q \left| f(x) - a_n^{(p)} x^p - a_n^{(p-1)} x^{p-1} - \cdots - a_n^{(0)} \right| dx = 0.$$

a) Prove that the sequences $a_n^{(p)}, a_n^{(p-1)}, \ldots, a_n^{(0)}$ are convergent. b) Prove that there exist real numbers $l_0, l_1, \ldots, l_p$ such that $f(x) = l_0 + l_1 x + \cdots + l_p x^p$.

347. Let $f : [0,1] \to \mathbb{R}$ be a C^1 function such that $f(0) = f(1) = 0$. Prove that

$$\left(\int_0^1 x^2 f(x)\, dx\right)^2 \leq \frac{1}{112} \int_0^1 (f'(x))^2\, dx.$$

348. Let $f : (0,\infty) \to (0,\infty)$ be an increasing function such that

$$\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = 1.$$

Prove that $\displaystyle\lim_{x\to\infty} \frac{f(x)}{a^x} = 0$, for any $a > 1$.

349. Let $f : [0,1] \to [0,1]$ be a continuous function which is derivable on $(0,1]$, such that $f'(x) < 0$ for $x \in (0,1]$, $f(1) = 0$ and $f'(1) < 0$. Prove that for every integer $n \geq 1$ the equation $f(x) = x^n$ has a unique solution in the interval $(0,1)$, denoted by a_n, and that

$$\lim_{n\to\infty} \frac{n}{\ln n}(a_n - 1) = -1.$$

350. Let $P, Q \in \mathbb{R}[X]$ be nonconstant monic polynomials such that $k = \deg P = \deg Q + 1$.

a) Prove that

$$\lim_{x\to\infty} \int_x^{2x} \frac{t^{k-1}}{P(t)}\, dt = \ln 2.$$

b) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\displaystyle\lim_{x\to\infty} \frac{f(x)}{x} = 1$, compute the limit

$$\lim_{x\to\infty} \int_x^{2x} \frac{f(Q(t))}{P(f(t))}\, dt.$$

351. Let $f:[0,1]\to\mathbb{R}$ be a differentiable function with continuous derivative such that $f(0)=f(1)$ and $\int_0^1 xf(x)\,dx=0$. Prove that

$$\int_0^1 (f'(x))^2\,dx \geq 180\left(\int_0^1 f(x)\,dx\right)^2.$$

352. Let $f\in C^{2n+1}([0,1],\mathbb{R})$ for some $n\geq 0$ such that $f(1/2)=f'(1/2)=f''(1/2)=\cdots=f^{(2n)}(1/2)=0$. Prove that

$$\int_0^1 \left(f^{(2n+1)}(x)\right)^2 dx-\left(\int_0^1 f^{(2n+1)}(x)\,dx\right)^2 \geq 2^{4n+2}(4n+3)\big((2n+1)!\big)^2\left(\int_0^1 f(x)\,dx\right)^2.$$

353. Prove that for all natural numbers $n\geq 4$ the equation

$$\sum_{k=1}^{n}\frac{1}{k^2+2k+x}=\frac{3}{4}$$

has a unique solution in the interval $(-1,0)$, denoted by x_n, and find the value of $\lim\limits_{n\to\infty} nx_n$.

354. Let $f:[0,1]\to\mathbb{R}$ be a differentiable function with continuous derivative on $[0,1]$ such that $f(0)=f(1/2)=f(1)=0$. Show that

$$\int_0^1 (f'(x))^2\,dx \geq 48\left(\int_0^1 f(x)\,dx\right)^2.$$

355. Find all C^2 functions $f:\mathbb{R}\to(0,\infty)$ such that there exists $\alpha>1$ for which

$$f''(x)f(x)\geq \alpha\big(f'(x)\big)^2,\qquad \forall x\in\mathbb{R}.$$

356. Let f be a real-valued function, at least five times continuously differentiable on the closed interval $[a,b]$, such that

$$f(a)=f(b)\qquad\text{and}\qquad f\left(\frac{a+3b}{4}\right)=f\left(\frac{3a+b}{4}\right).$$

Show that

$$\left|\int_a^{\frac{a+b}{2}} f(x)\,dx - \int_{\frac{a+b}{2}}^{b} f(x)\,dx\right| \leq \frac{(b-a)^6}{184320} \sup_{x\in[a,b]} \left|f^{(5)}(x)\right|,$$

where $f^{(5)}$ denotes the fifth derivative of f.

357. Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function, of class C^{2025}, for which $f^{(2k+1)}(0) = 0$ for all $k \in \overline{1, 1011}$, and $f^{(2025)}(0) > 0$.

a) Prove that there exists $\varepsilon > 0$ such that the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \ln\left(1 + \frac{1}{n^{f(x)}}\right)$$

is convergent for all $x \in (0, \varepsilon)$.

b) For all $x \in (0, \varepsilon)$, denote the sum of the previous series by $S(x)$ and let $g(x) = \dfrac{1}{S(x)}$. Study the convergence of the series

$$\sum_{n=1}^{\infty} g\left(\frac{1}{n^{\alpha}}\right),$$

where $\alpha > 0$.

358. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such that

$$\lim_{n\to\infty} a_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} (a_1 + a_2 + \cdots + a_n - na_{n+1}) = \ell \in \mathbb{R}.$$

Prove that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges and determine its sum.

359. Let $f : [0, \infty) \to \mathbb{R}$ be a function that admits the first-order derivative at 0 and such that $f(0) = 0$. Compute the value of

$$\lim_{n\to\infty} \frac{1}{n} \int_{2^n}^{2^{n+1}} f\left(\frac{\ln x}{x}\right) dx.$$

360. Let $(a_n)_{n\geq 1}$ be a sequence of nonnegative real numbers which converges to $a \in \mathbb{R}$.

a). Calculate

$$\lim_{n\to\infty} \sqrt[n]{\int_0^1 (1 + a_n x^n)^n \, dx}.$$

b). Calculate

$$\lim_{n\to\infty} \sqrt[n]{\int_0^1 \left(1 + \frac{a_1 x + a_3 x^3 + \cdots + a_{2n-1} x^{2n-1}}{n}\right)^n dx}.$$

361. Let $(a_n)_{n\geq 1}$ be a positive real sequence such that

$$\lim_{n\to\infty} \frac{a_n}{n} = a \in \mathbb{R}_+^* \quad \text{and} \quad \lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right)^n = b \in \mathbb{R}_+^*.$$

Compute

$$\lim_{n\to\infty} (a_{n+1} - a_n).$$

362. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be positive real sequences such that

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{n} = a \in \mathbb{R}_+^* \quad \text{and} \quad \lim_{n\to\infty} \frac{b_{n+1}}{n b_n} = b \in \mathbb{R}_+^*.$$

Compute

$$\lim_{n\to\infty} \left(\frac{a_{n+1}}{\sqrt[n+1]{b_{n+1}}} - \frac{a_n}{\sqrt[n]{b_n}}\right).$$

363. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I such that f'' is bounded on $[a,b]$, where $a, b \in I$, $a < b$. Prove that

$$\left|\frac{1}{b-a}\int_a^b f(u)\,du - \frac{1}{2}\left(\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right) + \frac{b-a}{48}\big(f'(b) - f'(a)\big)\right| \leq \frac{\sqrt{3}}{216}(b-a)^2 \|f''\|_{+\infty}.$$

364. Determine the biggest real number α, with the property that for any function $f : [0,1] \to [0,\infty)$ which meets the requirements:

i). f is convex and $f(0) = 0$;

ii). there exists $\varepsilon \in (0,1)$ such that f is differentiable on $[0,\varepsilon]$ and $f'(0) \neq 0$;

the inequality holds

$$\int_0^1 \left[\frac{x^2}{x\int_0^1 f(t)\,dt}\right] dx \geq \int_0^1 \left[(x+\alpha)\cdot\frac{f(x)}{\left(\int_0^1 f(t)\,dt\right)^2}\right] dx.$$

365. For every $n \geq 1$ define

$$x_n = \int_0^1 \ln(1+x+x^2+\cdots+x^n)\cdot\ln\frac{1}{1-x}\,dx.$$

(a) Show that x_n is finite for every $n \geq 1$ and $\lim\limits_{n\to\infty} x_n = 2$.

(b) Calculate $\lim\limits_{n\to\infty} \dfrac{n}{\ln n}(2-x_n)$.

366. Let $(x_n)_{n\geq 1}$ be the sequence defined by $x_1 \in (0,1)$ and $x_{n+1} = x_n - \dfrac{x_n^2}{2^n}$ for all $n \geq 1$. Prove that the sequence $(x_n)_{n\geq 1}$ is convergent to a limit $C > 0$ and, moreover,

$$\lim_{n\to\infty} 8^{n-1}\left(x_n - C - \frac{C^2}{2^{n-1}} - \frac{C^3}{3\cdot 4^{n-2}}\right) = \frac{12C^4+32C^3}{21}.$$

367. Let $f:[0,1]\to\mathbb{R}$ be a differentiable function with continuous derivative such that $f(1)=0$ and $f'(1)=1$. Prove that there exists $c\in(0,1)$ such that

$$f(c) = f'(c)\int_0^c f(x)\,dx.$$

368. Let $f:[0,1]\to\mathbb{R}$ be a continuous function such that

$$\int_0^1 x^k f(x)\,dx = 0 \quad \text{for } 0\leq k\leq n-1, \qquad \int_0^1 x^n f(x)\,dx = 1.$$

Prove that

$$\int_0^1 f^2(x)\,dx \geq (2n+1)\binom{2n}{n}^2.$$

369. Let $f:[0,1]\to\mathbb{R}$ be a differentiable function with continuous derivative such that $f(0)=f(1)=0$ and $\int_0^1 xf(x)\,dx = 0$. Prove that

$$\int_0^1 (f'(x))^2\,dx \geq 192\left(\int_0^1 f(x)\,dx\right)^2.$$

370. We consider the function f of class C^2 such that $f'(0) \neq 0$ and we denote by $c_n \in \left[0, \frac{1}{n}\right]$ the point for which

$$n \int_0^{1/n} f(x)\, dx = f(c_n).$$

Compute

$$\lim_{n\to\infty} n\left(nc_n - \frac{1}{2}\right).$$

371. Prove that there exists a maximum value of $\alpha \in \mathbb{R}$, and find that value, for which there is a sequence $(a_n)_{n\geq 1}$ of positive real numbers such that

$$\sum_{n=1}^{\infty} a_n^2 \text{ is convergent} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{a_n}{n^\alpha} \text{ is divergent.}$$

372. Find all continuous bijective functions

$$f : [0,1] \to [0,1]$$

such that

$$\int_0^1 g(f(x))\, dx = \int_0^1 g(x)\, dx$$

for every continuous function

$$g : [0,1] \to \mathbb{R}.$$

373. For $c > 0$ and $n \geq 1$, let

$$R_n(c) = \sqrt{c + \sqrt{c + \sqrt{c + \cdots + \sqrt{c}}}}$$

with n radicals. Evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \prod_{j=k}^{n} \frac{1}{R_j(c)}.$$

374. Let $f : [0,1] \to \mathbb{R}$ be a function of class C^1. Show that

$$f\left(\frac{1}{2}\right) \leq \int_0^1 |f(x)|\, dx + \frac{1}{2}\int_0^1 |f'(x)|\, dx.$$

375. Show that there is no differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x)| < 2x \quad \text{and} \quad f(x)f'(x) \geq \sin x, \qquad \text{for all } x \in \mathbb{R}.$$

376. Let $f(x)$ be a continuous real-valued function on $[0, \infty)$, differentiable on $(0, \infty)$, and satisfying $f(0) \geq 0$ and $f'(x) \geq (f(x))^3$ for $x > 0$. Prove that $f(x) = 0$ for all $x \geq 0$.

377. What is the maximum value of

$$\int_0^1 x^3 f(x)^2 \, dx - \int_0^1 x^2 f(x)^3 \, dx$$

over all continuous functions $f : [0, 1] \to [0, \infty)$?

378. Let $f : [0, 1] \to \mathbb{R}$ be continuous on $[0, 1]$ and differentiable on $(0, 1)$, and assume that either $f(0) = f(1)$ or $f(1)$ is not an extremal value for f.

(a) Prove that for any real λ except $\lambda = 1$, there are distinct $c_1, c_2 \in (0, 1)$ such that $f'(c_1) = \lambda f'(c_2)$.

(b) Prove that, whenever $0 \leq s_1 < s_2 < s_3 < s_4$, there are distinct $c_1, c_2, c_3, c_4 \in (0, 1)$ such that

$$s_1 f'(c_1) + s_2 f'(c_2) + s_3 f'(c_3) + s_4 f'(c_4) = 0.$$

379. Determine the functions $f : [0, 1] \to (0, \infty)$ of class C^1 which satisfy the relations

$$f(1) = ef(0) \qquad \text{and} \qquad \int_0^1 (f'(x))^2 \, dx + \int_0^1 \frac{1}{(f(x))^2} \, dx \leq 2.$$

380. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Show that the equation

$$2x - \int_0^x f(t) \, dt = 1$$

has a unique solution in $[0, 1]$.

381. Let $f : [a, b] \to \mathbb{R}$ be continuously differentiable with $f(a) = 0$. Suppose that for some $\lambda > 0$, it holds that $|f'(x)| \leq \lambda |f(x)|$ for all $x \in [a, b]$. Prove that $f = 0$.

382. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that for every $x \in [0, 1]$, it holds that

$$\int_x^1 f(t) \, dt \geq \frac{1 - x^2}{2}.$$

Show that $\displaystyle\int_0^1 f(t)^2 \, dt \geq \frac{1}{3}$.

383. Let $f : [0,1] \to \mathbb{R}$ be a continuous function with the property that for any $x, y \in [0,1]$, it holds that

$$xf(y) + yf(x) \leq 1.$$

Show that $\displaystyle\int_0^1 f(x)\,dx \leq \frac{\pi}{4}$. Can equality hold?

384. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$\left| \sum_{k=1}^{n} 3^k \left(f(x+ky) - f(x-ky) \right) \right| \leq 1,$$

for all $n \in \mathbb{N}$ and all $x, y \in \mathbb{R}$. Prove that f must be a constant function.

385. Find all strictly monotonous functions $f : (0,\infty) \to (0,\infty)$ satisfying

$$f(x^2/f(x)) = x \qquad \text{for all } x > 0.$$

386. For each positive integer n, let $f_n(\theta) = \sin(\theta)\sin(2\theta)\sin(4\theta)\cdots\sin(2^n\theta)$. Show that

$$|f_n(\theta)| \leq \frac{2}{\sqrt{3}}|f_n(\pi/3)| \qquad \text{for all } \theta \in \mathbb{R} \text{ and } n \in \mathbb{N}.$$

387. Let f be a twice differentiable function on $[0,1]$ such that $\displaystyle\int_0^1 f(x)\,dx = \frac{f(1)}{2}$. Prove that

$$\int_0^1 (f''(x))^2\,dx \geq 30\,(f(0))^2.$$

388. Let $0 < a < b$, $m = \dfrac{a+b}{2}$, and let $f : [a,b] \to \mathbb{R}$ be differentiable with continuous derivative on $[a,b]$, such that $f(m) = 0$. Prove that

$$2a^3 \int_{-a}^{a} (f'(x))^2\,dx \geq 3\left(\int_{-a}^{a} f(x)\,dx \right)^2.$$

389. Find all continuous real functions f on $[-1,1]$ that satisfy the integral equation

$$x^2 + \int_1^{1/x} f(x^2 t)\,dt = 1.$$

390. Find all non-constant continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{1}{y-x}\int_x^y f(g(t))\,dt = f\left(\frac{x+y}{2}\right), \qquad \forall x, y \in \mathbb{R},\ x \neq y.$$

391. Let $f : [0,1] \to \mathbb{R}$ be continuous, with $\displaystyle\int_0^1 x(x-1)^2 f(x)\,dx = 0$. Prove that there is a real number $c \in (0,1)$ such that

$$\int_0^c x^2 f(x)\,dx = c\int_0^c x f(x)\,dx.$$

392. Define the sequence $\{a_n\}$ by the recursion $a_1 = 1$, $a_{n+1} = a_n + \dfrac{1}{q \cdot a_n}$ for $n > 0$, $q > 0$. Find the constant $c(q)$ such that

$$\lim_{n\to\infty}\left(a_n - \sqrt{c(q)\cdot n}\right) = 0.$$

393. Let $0 < a < b$ and let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b), with $f(a) = f(b)$. Prove that there exist distinct $c_1, c_2 \in (a,b)$ such that

$$\sqrt{b}\,f'(c_1) + \sqrt{a}\,f'(c_2) = 0.$$

394. Let I be an open interval containing 0 and 1, and let $f : I \to \mathbb{R}$ be a differentiable, strictly increasing, convex function. If $f'(1) < 2f(1)$, prove that there exist positive real numbers a, b such that

$$\int_0^1 (f(x))^{2n+1}\,dx \sim a\cdot\frac{b^n}{n} \qquad \text{as } n \to \infty,$$

and express a and b as functions of $f(1)$ and $f'(1)$.

395. Let $f : [0,1] \to [0,\infty)$ be a continuous function with $\displaystyle\int_0^1 x f(x)\,dx = 1$. Prove that

$$\int_0^1 f(x)\,dx + \int_0^1 \sqrt{f(x)}\,dx + \int_0^1 f^2(x)\sqrt{f(x)}\,dx \geq \frac{19}{5}.$$

396. Let $f : [0,1] \to [0,\infty)$ be continuous. Show that

$$\int_0^1 f^5(x)\,dx \geq 6\left(\int_0^1 x^3 f^2(x)\,dx\right)\left(\int_0^1 x^2 f^3(x)\,dx\right),$$

where $f^n(x) = (f(x))^n$ when $n \geq 1$.

397. Consider for $x \in (0,1]$ and $n \in \mathbb{N} := \{0,1,2,\dots\}$ the equation

$$\frac{(1-x)^{n-1}}{x^{2n-1}} = 2. \tag{$*$}$$

Prove the following:

(1) For each n there exists a unique $x_n \in (0,1)$ solving equation $(*)$.

(2) Prove that $L := \lim_{n\to\infty} x_n$ exists.

(3) Determine L.

398. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $\displaystyle\int_0^1 f(t)\,dt = 1$, and let n be a positive integer. Show that

1. there are distinct $c_1, c_2, \dots, c_n$ in $(0,1)$ such that

$$f(c_1) + f(c_2) + \cdots + f(c_n) = n,$$

2. there are distinct $c_1, c_2, \dots, c_n$ in $(0,1)$ such that

$$\frac{1}{f(c_1)} + \frac{1}{f(c_2)} + \cdots + \frac{1}{f(c_n)} = n.$$

399. Suppose that $f : [0,1] \to \mathbb{R}$ has a continuous second derivative with $f''(x) > 0$ on $(0,1)$, and suppose that $f(0) = 0$. Choose $a \in (0,1)$ such that $f'(a) < f(1)$. Show that there is a unique $b \in (a,1)$ such that

$$f'(a) = \frac{f(b)}{b}.$$

400. Let f, g be two differentiable real functions such that $g(x) \neq 0$ for all real numbers x. Suppose that c is a real number such that

$$f(c)\int_a^b g(x)\,dx \neq g(c)\int_a^b f(x)\,dx,$$

for all pairwise distinct real numbers a and b. Prove that $(f/g)'(c) = 0$.

401. What is the minimum value of $\displaystyle\int_0^1 (f'(x))^2\,dx$ over all continuously differentiable functions $f:[0,1]\to\mathbb{R}$ such that

$$\int_0^1 f(x)\,dx = \int_0^1 x^2 f(x)\,dx = 1\,?$$

402. Find all continuous functions $f:[0,\infty)\to\mathbb{R}$ such that, for all positive x,

$$f(x)\left(f(x)-\frac{1}{x}\int_0^x f(t)\,dt\right)\geq (f(x)-1)^2.$$

403. Let $g:[0,1]\to\mathbb{R}$ be continuous. Prove that

$$\lim_{n\to\infty}\frac{n}{2^n}\int_0^1 \frac{g(x)}{x^n+(1-x)^n}\,dx = C\,g(1/2)$$

for some constant C (independent of g), and determine the value of C.

404. Let $(x_n)_{n\geq 0}$ be a sequence of complex numbers defined recursively by

$$x_{n+1}=\frac{1}{k}\big(x_n+x_{n-1}+x_{n-2}+\cdots+x_{n-k+1}\big),\qquad n\geq k-1.$$

Determine $\lim_{n\to\infty} x_n$.

405. Let $p\in\mathbb{N}$, $p\geq 2$, and the sequence $(x_n)_{n\geq 1}$

$$x_n=\frac{1}{n}\cdot\sqrt[p]{\sum_{k=1}^{n}(k^p-(k-1)^p)\left(\frac{k-1}{k}\right)^p}$$

for any $n\in\mathbb{N}^*$.

a) Prove that $x_n\in[0,1]$ and $x_n\leq x_{n+1}$ for any $n\in\mathbb{N}^*$.

b) Prove that $\lim_{n\to\infty} x_n = 1$ and calculate $\lim_{n\to\infty}\big(nx_n^p-(n-1)x_{n-1}^p\big)$.

406. Prove that

$$\lim_{\varepsilon\to 0^+}\left(\frac{1}{\varepsilon^2}-\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\int_{\varepsilon\sqrt{n}}^{\infty}e^{-t^2/2}\,dt\right)=\frac{1}{2}.$$

407. Suppose a is a given real number. Consider the real sequence $\{x_n\}$ given by

$$x_1 = b, \qquad x_{n+1} = x_n^2 + (1-2a)x_n + a^2, \qquad n \geq 1.$$

Determine b for which the sequence $\{x_n\}$ converges, and compute the limit of the sequence in that case.

408. Let $f(x)$ be a function defined and continuous on $[0,1]$ satisfying

$$\int_{x_1}^{x_2} [f(x)]^2 \, dx \leq \frac{x_2^3 - x_1^3}{3}$$

for all $x_1, x_2 \in [1,2]$ with $x_1 \leq x_2$. Prove that

$$\int_1^2 f(x) \, dx \leq \frac{3}{2}.$$

409. Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. Suppose that $|f''(x)| \leq 1$ for all x and that $f(-1) = f'(-1) = f(1) = f'(1) = 0$. What is the maximum possible value for $f(0)$? Justify your answer.

410. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a differentiable function at the origin and satisfying $\varphi(0) = 0$. Evaluate

$$\lim_{x \to 0} \frac{1}{x} \left[\varphi(x) + \varphi\left(\frac{x}{2}\right) + \cdots + \varphi\left(\frac{x}{n}\right) \right],$$

where n is a positive integer.

411. Let $f : [0,1] \to \mathbb{R}$ be a function of class $C^1([0,1])$ satisfying $|f'(x)| \leq 1$ for $x \in [0,1]$ and

$$\left| \int_0^1 f(x) \, dx \right| \leq \frac{1}{2}.$$

Prove that

$$\left| \int_0^1 x^n f(x) \, dx \right| \leq \frac{1}{n+2}, \qquad n \geq 1.$$

412. Let f_n, $n = 1, 2, \ldots$, be the sequence of functions on $(0, \infty)$ defined by

$$f_n(x) = \frac{1}{n} \left(1 - \frac{x}{n}\right)^n e^x, \quad 0 < x < n, \qquad f_n(x) = 0, \quad x \geq n.$$

Prove that the sequence $a_n = \int_0^\infty f_n(x) \, dx$, $n = 1, 2, \ldots$, converges, and identify $a_\infty = \lim_{n \to \infty} a_n$.

413. Let $K = \{f : (0, +\infty) \to \mathbb{R} \mid \int_0^\infty f^4(x)\, dx \leq 1\}$. Evaluate

$$\sup_{f \in K} \int_0^\infty f^3(x) e^{-x}\, dx.$$

414. Let $f : [1, \infty) \to \mathbb{R}$ be bounded and continuous. Prove that

$$\lim_{n \to \infty} \int_1^\infty f(t)\, n t^{-n-1}\, dt = f(1).$$

415. Let $f : [0, 3] \to \mathbb{R}$ be continuous on $[0, 3]$ and differentiable on $(0, 3)$. Prove that there exists $c \in (0, 3)$ such that

$$f'(c) = \frac{f(3) - f(2) + f(1) - f(0)}{2}.$$

416. Let $f : \mathbb{R} \to \mathbb{R}$ have derivatives up to second order, with f'' continuous on $\mathbb{R}$. Moreover, suppose f satisfies

$$|f(x)| \leq 2024, \qquad \forall x \in \mathbb{R}.$$

Prove that there exists $x \in \mathbb{R}$ such that $f''(x) = 0$.

417. Let α be a positive real number, and let f be a continuous, decreasing function on $[0, 1]$, with $a \in (0, 1)$ satisfying

$$\int_0^a f(t)\, dt < \frac{a}{2025}$$

and $f(0) = \beta > 0$. Prove that the equation $f(x) = x^{2024}$ has a solution in $[0, 1]$.

418. Let f be differentiable on $[0, 1]$ satisfying $\int_0^1 f(x)\, dx = 0$. Prove that there exists $c \in (0, 1)$ such that

$$3f'(c) + 32 \int_0^{1/4} f(x)\, dx = 0.$$

419. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function with $f(1) = 0$. Prove the existence and determine the value of the limit

$$\lim_{\substack{t \to 1 \\ t < 1}} \left(\frac{1}{1-t} \int_0^1 x \big(f(tx) - f(x) \big)\, dx \right).$$

420. Let $x_1 = 1$ and
$$x_{n+1} = \frac{1}{x_n}\left(\sqrt{1+x_n^2}-1\right)$$
for $n \geq 1$. Show that the sequence $\{2^n x_n\}$ converges and find its limit.

421. Let $f : [a,b] \to \mathbb{R}$ be a differentiable function. Prove that for any positive integer n there are numbers $\theta_1 < \theta_2 < \cdots < \theta_n$ in the interval (a,b) such that
$$\frac{f(b)-f(a)}{b-a} = \frac{f'(\theta_1)+f'(\theta_2)+\cdots+f'(\theta_n)}{n}.$$

422. Solve for real numbers:
$$\sum_{i=1}^{p}\left(\sum_{j=1}^{q}\left(\log x - \frac{1}{i^5}\right)\left(\log x - \frac{1}{j^7}\right)\right) = 0; \quad p,q \in \mathbb{N}\setminus\{0\}$$

423. If $0 < a \leq b < \dfrac{\pi}{2}$ then:
$$\int_a^b \frac{|\sin x \cdot \sin(2x)\cdots\sin(2019x)|}{\sin^{2019} x}\,dx \leq 2019!\,(b-a)$$

424. $f : \mathbb{R} \to \mathbb{R}$, f-continuous, $a,b \in \mathbb{R}$, $a \leq b$. Prove that:
$$\int_a^b (f^8(x)+f^2(x))\,dx + b - a \geq \int_a^b (f^5(x)+f(x))\,dx$$

425. If $0 < a \leq b$, $f : [a,b] \to (0,\infty)$, f-continuous, then:
$$(b-a)\left(\int_a^b f^3(x)\,dx\right)\left(\int_a^b \frac{dx}{f^2(x)}\right) \geq \left(\int_a^b \sqrt[3]{f^5(x)}\,dx\right)\left(\int_a^b \frac{dx}{\sqrt[3]{f(x)}}\right)^2$$

426. Suppose that $f : [0,1] \to \mathbb{R}$ has a continuous third derivative and $f(0) = f(1)$. Prove that
$$\left|\int_0^1 f'(x)x^{k-1}(1-x)^{k-1}\,dx\right| \leq \frac{(k-1)k!(k-1)!}{6(2k+1)!}\max_{0\leq x\leq 1}|f'''(x)|,$$
where k is a positive integer.

427. Let f be a twice continuously differentiable function from $[0,1]$ to $\mathbb{R}$. Let p be an integer greater than 1. Given that

$$\sum_{k=1}^{p-1} f(k/p) = -\frac{1}{2}\big(f(0)+f(1)\big),$$

prove that

$$\left(\int_0^1 f(x)\,dx\right)^2 \leq \frac{1}{5!\,p^4}\int_0^1 (f''(x))^2\,dx.$$

428. Let f be a continuously differentiable function on $[0,1]$. Let

$$A = f(1) \qquad \text{and} \qquad B = \int_0^1 x^{-1/2} f(x)\,dx.$$

Evaluate, in terms of A and B,

$$\lim_{n\to\infty} n\left(\int_0^1 f(x)\,dx - \sum_{k=1}^{n}\left(\frac{k^2}{n^2}-\frac{(k-1)^2}{n^2}\right) f\left(\frac{(k-1)^2}{n^2}\right)\right).$$

429. Let $f:[0,1]\to\mathbb{R}$ be a differentiable function with continuous derivative such that $f(1)=0$. Show that

$$4\int_0^1 x^2|f'(x)|^2\,dx \geq \int_0^1 |f(x)|^2\,dx + \left(\int_0^1 |f(x)|\,dx\right)^2.$$

430. Compute each of the following limits.

(a) Let

$$f(x) = \frac{ax+b}{cx+d},$$

where $a,b,c,d\in\mathbb{R}$, $ad-bc\neq 0$, and $f:(0,\infty)\to(0,\infty)$. Find:

$$\Omega = \lim_{n\to\infty}\left(\lim_{x\to 0^+}(\underbrace{f\circ f\circ\cdots\circ f}_{n\text{ times}})(x)\right).$$

(b)

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{3^n}\sqrt[n]{\frac{1}{n!}\prod_{i=1}^{n}\left(\sum_{k=1}^{i} k\binom{2i}{2k}\right)}\right)$$

(c)

$$\Omega(p) = \lim_{n\to\infty} \left(\frac{1}{p} \sum_{k=1}^{n} \sqrt[n]{k} \right)^{np}, \quad p \in \mathbb{N} - \{0\}$$

Find:

$$\Omega = \sum_{k=1}^{\infty} \frac{1}{\Omega(p)}$$

(d)

$$\Omega = \lim_{n\to\infty} \left(n^6 \sin\frac{1}{n^3} \tan\frac{1}{n^5} \sum_{1\le k<l\le n} \sin\left(\frac{k+l}{n}\right) \right)$$

431. Compute each of the following limits.

(a)

$$\Omega = \prod_{n=1}^{\infty} \left(1 + \left(\frac{1}{\pi}\right)^{3^n} + \left(\frac{1}{\pi^2}\right)^{3^n} \right)$$

(b)

$$\Omega = \lim_{n\to\infty} \left(n \left(\frac{\log\left(1 + \frac{\sqrt[n]{e}}{n}\right)^{n+1}}{\log\left(1 + \frac{\sqrt[n+1]{e}}{n+1}\right)^{n}} - 1 \right) \right)$$

(c)

$$\Omega = \prod_{k=1}^{\infty} \left(\lim_{n\to\infty} \left(\frac{\sqrt[n]{k+1} + \sqrt[n]{k+3}}{\sqrt[n]{k+2} + \sqrt[n]{k+4}} \right)^{n} \right)^{2}$$

(d)

$$\Omega = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{k \cdot (2n-2k-1)!!}{(k+1)! \cdot (2+2n-2k)!!} \right)$$

432. Compute each of the following limits.

(a)

$$\Omega_k(m) = 2 \lim_{x\to 0} \left(\frac{1 - (\cos kx)^{\frac{1}{k^{m+2}}}}{x^2} \right), \quad k, m \in \mathbb{N}^*$$

Find a closed form for:

$$\Omega = \left(\sum_{k=1}^{\infty} \Omega_k(2) \right) \left(\sum_{k=1}^{\infty} \Omega_k(3) \right)$$

(b)

$$\Omega = \lim_{n\to\infty} \left(\sqrt{\frac{2n+2}{n^3 \cdot 2^n (2^{n+1} - n)}} \sum_{k=1}^{n} \sqrt{\frac{k}{k+1}} \binom{n}{k} \right)$$

(c)

$$\Omega = \lim_{n\to\infty} \left(\frac{1}{\sqrt{2n-1}} \sum_{k=1}^{n} \frac{1}{\sqrt{(2k-1)k!}} \right)$$

433. Compute each of the following limits.

(a)

$$\Omega = \lim_{n\to\infty} \left(n \sum_{k=1}^{n} \frac{k(k+1)}{(k^2+n^2)(k^2+2k+1+n^2)} \right)$$

(b)

$$\Omega(n) = \lim_{x\to 0} \left(\frac{5^x - 1}{x^{n+1}} - \frac{\log 5}{x^n} - \frac{\log^2 5}{2x^{n-2}} - \cdots - \frac{\log^n 5}{n! \cdot x} \right), \quad n \in \mathbb{N},\ n \geq 2$$

(c)

$$\Omega = \lim_{n\to\infty} \frac{\sqrt{(1+n!)^{n!}}}{n \cdot (n!)!}$$

434. Compute each of the following limits.

(a)

$$\Omega = \lim_{n\to\infty} \left((1-i) \prod_{k=1}^{n} \frac{k^2+k+1+i}{\sqrt{(k^2+1)(k^2+2k+2)}} \right)$$

(b)

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\left(\sum_{k=1}^{n} k^3 \binom{n}{k}^2 \right) \left(\sum_{k=1}^{n} k^2 \binom{n}{k}^2 \right)^{-1}}$$

(c)

$$\Omega = \lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=0}^{n} \sum_{i=0}^{k} \sum_{j=0}^{i} (-1)^j \cdot \binom{i}{j} \cdot \frac{3^{i-j}}{4^i} \right)$$

(d)

$$\Omega = \lim_{n\to\infty} \left(\frac{1}{2} \left(1 + \frac{n^{1/n}}{n} \right)^{1/n} + \frac{1}{2} \left(1 - \frac{n^{1/n}}{n} \right)^{1/n} \right)^n$$

(e)

$$\Omega = \lim_{n\to\infty} \left(n^{n-2} \cdot \left(\frac{2}{3}\right)^2 \cdot \left(\frac{3}{5}\right)^3 \cdots \left(\frac{n}{2n-1}\right)^n \right)$$

435. Let n be a nonzero natural number and $f : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ be a function such that $f(2014) = 1 - f(2013)$. Let $x_1, x_2, x_3, \ldots, x_n$ be real numbers not equal to each other. If

$$\begin{vmatrix} 1+f(x_1) & f(x_2) & f(x_3) & \cdots & f(x_n) \\ f(x_1) & 1+f(x_2) & f(x_3) & \cdots & f(x_n) \\ f(x_1) & f(x_2) & 1+f(x_3) & \cdots & f(x_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f(x_1) & f(x_2) & f(x_3) & \cdots & 1+f(x_n) \end{vmatrix} = 0,$$

prove that f is not continuous.

436. Consider the sequence (x_n) given by

$$x_1 = 2, \qquad x_{n+1} = \frac{x_n + 1 + \sqrt{x_n^2 + 2x_n + 5}}{2}, \quad n \geq 2.$$

Prove that the sequence $y_n = \displaystyle\sum_{k=1}^{n} \frac{1}{x_k^2 - 1}$, $n \geq 1$ is convergent and find its limit.

437. Prove that for any sequence of nonnegative numbers (a_n),

$$\liminf_{n\to\infty} \left(n^2 \left(4a_n(1 - a_{n-1}) - 1\right)\right) \leq \frac{1}{4}.$$

438. Evaluate

$$\int_a^b \left\{ \frac{x^2+1}{x^2-x+1} \right\} dx$$

in terms of a and b, where $a < 0 < b$ and $\{t\}$ denotes the fractional part of t.

439. Let

$$\mathcal{A} = \left\{ f \in \mathcal{R}([0,1]) : \int_0^1 f(x)\,dx = 3, \int_0^1 xf(x)\,dx = 2 \right\}.$$

Find

$$\min_{f\in\mathcal{A}} \int_0^1 f^2(x)\,dx$$

and a function for which the minimum is attained.

440. Let

$$\mathcal{A} = \{f \in C^2([0,1]) : f(0) = f(1) = 0,\ f'(0) = a\}.$$

Find

$$\min_{f\in\mathcal{A}} \int_0^1 (f''(x))^2\,dx$$

and a function for which the minimum is attained.

441. Does there exist a function f continuously differentiable on $[0,2]$ and such that $f(0) = f(2) = 1$, $|f'(x)| \leq 1$ for $x \in [0,2]$ and $\left|\int_0^2 f(x)\,dx\right| \leq 1$?

442. Suppose f is continuously differentiable on $[a,b]$ and $f(a) = f(b) = 0$. Show that

$$\max_{x\in[a,b]} |f'(x)| \geq \frac{4}{(b-a)^2} \int_a^b |f(x)|\,dx.$$

443. Prove the following *Hölder inequality*: If functions $f_1, f_2, \ldots, f_n$ are nonnegative and Riemann integrable on $[a,b]$, and positive real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ satisfy $\sum_{i=1}^n \alpha_i = 1$, then

$$\int_a^b \prod_{i=1}^n f_i^{\alpha_i}(x)\,dx \leq \prod_{i=1}^n \left(\int_a^b f_i(x)\,dx\right)^{\alpha_i}.$$

444. Suppose that f and g are nonnegative and Riemann integrable on $[a,b]$. For $p \neq 0$ let q be its conjugate, that is, $\frac{1}{p} + \frac{1}{q} = 1$. Using the previous result, prove that

(a) if $p > 1$, then

$$\int_a^b f(x)g(x)\,dx \leq \left(\int_a^b f^p(x)\,dx\right)^{1/p} \left(\int_a^b g^q(x)\,dx\right)^{1/q},$$

(b) if $p < 0$ or $0 < p < 1$, and f, g are positive, then

$$\int_a^b f(x)g(x)\,dx \geq \left(\int_a^b f^p(x)\,dx\right)^{1/p} \left(\int_a^b g^q(x)\,dx\right)^{1/q}.$$

445. Assume that $f_1, f_2, \ldots, f_n$ are nonnegative and Riemann integrable on $[a,b]$.

(a) If $k > 1$, then

$$\int_a^b (f_1(x) + \cdots + f_n(x))^k\,dx \geq \int_a^b f_1^k(x)\,dx + \cdots + \int_a^b f_n^k(x)\,dx.$$

(b) If $0 < k < 1$, then

$$\int_a^b (f_1(x) + \cdots + f_n(x))^k \, dx \le \int_a^b f_1^k(x) \, dx + \cdots + \int_a^b f_n^k(x) \, dx.$$

446. Prove the following inequality: If g is Riemann integrable on $[a, b]$ and $0 \le g(x) \le 1$ for every $x \in [a, b]$, and f decreases on that interval, then

$$\int_{b-\lambda}^b f(x) \, dx \le \int_a^b f(x)g(x) \, dx \le \int_a^{a+\lambda} f(x) \, dx,$$

where

$$\lambda = \int_a^b g(x) \, dx.$$

447. Prove the following Bellman generalization of Steffensen's inequality: If g is Riemann integrable on $[0, 1]$ and $0 \le g(x) \le 1$ for every $x \in [0, 1]$, and f is nonnegative and decreasing on that interval, then for $p \ge 1$

$$\left(\int_0^1 f(x)g(x) \, dx \right)^p \le \int_0^\lambda (f(x))^p \, dx,$$

where

$$\lambda = \left(\int_0^1 g(x) \, dx \right)^p.$$

448. Prove the following generalization of Jensen's inequality: Suppose that f and p are Riemann integrable on $[a, b]$, $m \le f(x) \le M$, $p(x) \ge 0$ and $\int_a^b p(x) \, dx > 0$. If φ is continuous and convex on $[m, M]$, then

$$\varphi \left(\frac{1}{\int_a^b p(x) \, dx} \int_a^b p(x)f(x) \, dx \right) \le \frac{1}{\int_a^b p(x) \, dx} \int_a^b p(x)\varphi(f(x)) \, dx.$$

449. Let f be Riemann integrable on $[0, 1]$ and $|f(x)| \le 1$, $x \in [0, 1]$. Show that

$$\int_0^1 \sqrt{1 - f^2(x)} \, dx \le \sqrt{1 - \left(\int_0^1 f(x) \, dx \right)^2}.$$

450. Let f be nonnegative and monotone on $[0, 1]$, and let $a, b \ge 0$.

(a) If f is decreasing, prove that

$$\left(1-\left(\frac{a-b}{a+b+1}\right)^2\right)\int_0^1 x^{2a}f(x)\,dx\int_0^1 x^{2b}f(x)\,dx \geq \left(\int_0^1 x^{a+b}f(x)\,dx\right)^2.$$

(b) If f is increasing, prove the reverse inequality

$$\left(1-\left(\frac{a-b}{a+b+1}\right)^2\right)\int_0^1 x^{2a}f(x)\,dx\int_0^1 x^{2b}f(x)\,dx \leq \left(\int_0^1 x^{a+b}f(x)\,dx\right)^2.$$

451. Let f be continuous on $[a,b]$ and put $f(x)=0$ for $x \notin [a,b]$. For $h>0$, we define f_h by setting

$$f_h(x)=\frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt.$$

Show that

$$\int_a^b |f_h(x)|\,dx \leq \int_a^b |f(x)|\,dx.$$

452. Prove the *Minkowski inequality* for integrals. Assume that $f_1, f_2, \ldots, f_n$ are nonnegative and Riemann integrable on $[a,b]$.

(a) If $k>1$, then

$$\left(\int_a^b (f_1(x)+\cdots+f_n(x))^k\,dx\right)^{1/k} \leq \left(\int_a^b f_1^k(x)\,dx\right)^{1/k}+\cdots+\left(\int_a^b f_n^k(x)\,dx\right)^{1/k}.$$

(b) If $0<k<1$ and f, g are positive, then

$$\left(\int_a^b (f_1(x)+\cdots+f_n(x))^k\,dx\right)^{1/k} \geq \left(\int_a^b f_1^k(x)\,dx\right)^{1/k}+\cdots+\left(\int_a^b f_n^k(x)\,dx\right)^{1/k}.$$

453. Use the Steffensen inequality to prove that if f is continuously differentiable on $[a, b]$ and $m \le f'(x) \le M$ $(m < M)$ for $x \in [a, b]$, then

$$m + \frac{(M-m)\lambda^2}{(b-a)^2} \le \frac{f(b)-f(a)}{b-a} \le M - \frac{(M-m)(b-a-\lambda)^2}{(b-a)^2},$$

where

$$\lambda = \frac{f(b)-f(a)-m(b-a)}{M-m}.$$

454. Prove that if f is continuously differentiable on $[a, b]$ and $m \le f'(x) \le M$ $(m < M)$ for $x \in [a, b]$, then

$$\left| \int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}(b-a) \right| \le \frac{(f(b)-f(a)-m(b-a))(M(b-a)-f(b)+f(a))}{2(M-m)}.$$

455. Prove the following *Opial inequality*: If f is continuously differentiable on $[0, a]$ and $f(0) = 0$, then

$$\int_0^a |f(x)f'(x)|\,dx \le \frac{a}{2}\int_0^a (f'(x))^2\,dx.$$

456. Prove the following generalization of Opial's inequality. If $f \in C^n([0, a])$ is such that $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$, $n \ge 1$, then

$$\int_0^a |f(x)f^{(n-1)}(x)|\,dx \le \frac{a^n}{2}\int_0^a (f^{(n)}(x))^2\,dx.$$

457. Prove this generalization of Opial's inequality. If f is continuously differentiable on $[0, a]$, $f(0) = 0$ and $p \ge 0, q \ge 1$, then

$$\int_0^a |f(x)|^p |f'(x)|^q\,dx \le \frac{qa^p}{p+q}\int_0^a |f'(x)|^{p+q}\,dx.$$

458. Suppose $f \in C^{2n}([a, b])$ is such that $f^{(k)}(a) = f^{(k)}(b) = 0$ for $k = 0, 1, \ldots, n-1$. Prove that if $M = \max_{x\in[a,b]} |f^{(2n)}(x)|$, then

$$\left| \int_a^b f(x)\,dx \right| \le \frac{(n!)^2(b-a)^{2n+1}}{(2n)!(2n+1)!}M.$$

459. Let function $f(x)$ be defined on $\mathbb{R}$ and be infinitely differentiable. Given that $f(0) = f(101) \neq f(1)$ and for any x it holds true

$$f'''(x) + 3f''(x)f'(x) + (f'(x))^3 = 0.$$

Find the value of the following expression

$$\frac{|f(1)| + |f(2)| + \cdots + |f(100)|}{|f(1)| + |f(2)| + \cdots + |f(50)|}.$$

460. Let $f : (0, +\infty) \to \mathbb{R}$ and $f(2) \neq 0$. Given that for any positive numbers x, y it holds true

$$xf(x) - yf(y) = xy^2 f\left(\frac{x}{y}\right).$$

Find $\dfrac{4f(8)}{f(2)}$.

461. Let

$$x_n = \left(\frac{[n + \sin n]}{n + \sin n}\right)^{\frac{n+\sin n}{\{n+\sin n\}}}.$$

Evaluate the following expression

$$e \lim_{n\to\infty} x_n,$$

where $[x]$ is the integer part and $\{x\}$ is the fractional part of a real number x.

462. Evaluate the expression

$$\lim_{x\to+\infty} \left(\sin(\sin(\sin\sqrt{x^2+1})) - \sin(\sin(\sin x))\right).$$

463. Let $f_n : [0, 1] \to [0, \infty)$ be a continuous function, for $n = 1, 2, \ldots$. Suppose that one has

$$(*) \quad f_1(x) \geq f_2(x) \geq f_3(x) \geq \cdots \quad \text{for all } x \in [0, 1].$$

Let $f(x) = \lim_{n\to\infty} f_n(x)$ and $M = \sup_{0\leq x\leq 1} f(x)$.

1. Prove that there exists $t \in [0, 1]$ with $f(t) = M$.
2. Show by example that the conclusion of Part 1 need not hold if instead of $(*)$ we merely know that for each $x \in [0, 1]$ there exists n_x such that for all $n \geq n_x$ one has $f_n(x) \geq f_{n+1}(x)$.

464. Let $k \geq 0$ be an integer and define a sequence of maps

$$f_n : \mathbb{R} \to \mathbb{R}, \qquad f_n(x) = \frac{x^k}{x^2 + n}, \quad n = 1, 2, \ldots.$$

For which values of k does the sequence converge uniformly on $\mathbb{R}$? On every bounded subset of $\mathbb{R}$?

465. Let $f : [0, 1] \to \mathbb{R}$ be continuous and $k \in \mathbb{N}$. Prove that there is a real polynomial $P(x)$ of degree $\leq k$ which minimizes (for all such polynomials)

$$\sup_{0 \leq x \leq 1} |f(x) - P(x)|.$$

466. For each positive integer n, define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \cos nx$. Prove that the sequence of functions $\{f_n\}$ has no uniformly convergent subsequence.

467. Let $0 \leq a \leq 1$ be given. Determine all nonnegative continuous functions f on $[0, 1]$ which satisfy the following three conditions:

$$\int_0^1 f(x)\,dx = 1, \qquad \int_0^1 x f(x)\,dx = a, \qquad \int_0^1 x^2 f(x)\,dx = a^2.$$

468. Let f be a differentiable function on $[0, 1]$ and let

$$\sup_{0<x<1} |f'(x)| = M < \infty.$$

Let n be a positive integer. Prove that

$$\left| \sum_{j=0}^{n-1} \frac{f(j/n)}{n} - \int_0^1 f(x)\,dx \right| \leq \frac{M}{2n}.$$

469. Let $f : [0, \infty) \to \mathbb{R}$ be a uniformly continuous function with the property that

$$\lim_{b \to \infty} \int_0^b f(x)\,dx$$

exists (as a finite limit). Show that

$$\lim_{x \to \infty} f(x) = 0.$$

470. Let f be a real valued continuous function on $[0, \infty)$ such that

$$\lim_{x\to\infty}\left(f(x) + \int_0^x f(t)\,dt\right)$$

exists. Prove that

$$\lim_{x\to\infty} f(x) = 0.$$

471. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotone decreasing function, defined on the positive real numbers with

$$\int_0^\infty f(x)\,dx < \infty.$$

Show that

$$\lim_{x\to\infty} xf(x) = 0.$$

472. Let f be a continuous real valued function satisfying $f(x) \geq 0$, for all x, and

$$\int_0^\infty f(x)\,dx < \infty.$$

Prove that

$$\frac{1}{n}\int_0^n xf(x)\,dx \to 0$$

as $n \to \infty$.

473. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function given by

$$f(x) = \begin{cases} 1 & \text{if } x = 1, \\ e^{(x^{10}-1)} + (x-1)^2 \sin\left(\dfrac{1}{x-1}\right) & \text{if } x \neq 1. \end{cases}$$

(a) Find $f'(1)$.

(b) Evaluate $\displaystyle\lim_{u\to\infty}\left[100u - u\sum_{k=1}^{100} f\left(1 + \frac{k}{u}\right)\right]$.

474. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is differentiable at 0. Define another function $g : \mathbb{R} \to \mathbb{R}$ as follows:

$$g(x) = \begin{cases} f(x)\sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Suppose that g is also differentiable at 0. Prove that

$$g'(0) = f'(0) = f(0) = g(0) = 0.$$

475. Let $P(x)$ be a polynomial with real coefficients. Let $\alpha_1, \dots, \alpha_k$ be the distinct real roots of $P(x) = 0$. If P' is the derivative of P, show that for each $i = 1, 2, \dots, k$,

$$\lim_{x \to \alpha_i} \frac{(x - \alpha_i)P'(x)}{P(x)} = r_i,$$

for some positive integer r_i.

476. Let f and g be nonnegative continuous functions on $[0,1]$ with $f(0) = g(0) = 0$, and let $h : [0,1] \to \mathbb{R}$ be nondecreasing. Suppose that f is convex and g is concave. Prove

$$\int_0^1 g(x)\,dx \cdot \int_0^1 f(x)h(x)\,dx \geq \int_0^1 f(x)\,dx \cdot \int_0^1 g(x)h(x)\,dx.$$

477. Let $a_0 = \dfrac{1}{2}$ and a_n be defined inductively by

$$a_n = \sqrt{\frac{1 + a_{n-1}}{2}}, \qquad n \geq 1.$$

(a) Show that for $n = 0, 1, 2, \dots$, $a_n = \cos\theta_n$ for some $0 < \theta_n < \dfrac{\pi}{2}$, and determine θ_n.

(b) Using (a) or otherwise, calculate $\lim\limits_{n \to \infty} 4^n(1 - a_n)$.

478. For a function f suppose that f', f'' and f''' exist.

(a) Must the following limit exist? Justify your answer and if the limit exists, calculate it.

$$\lim_{n \to \infty} \left(n \int_0^1 x^n f(x)\,dx \right).$$

(b) Suppose the following limit is a real number a. Calculate $f(1)$.

$$\lim_{n \to \infty} \left(n^2 \int_0^1 x^n f(x)\,dx \right).$$

479. Let a be a positive real number, and let S_a be the set of functions $f : [-a, a] \to \mathbb{R}$ such that $\int_{-a}^{a} (f(x))^2\,dx = 1$. Let

$$A(f) = \int_{-a}^{a} f(x)\,dx, \qquad B(f) = \int_{-a}^{a} x f(x)\,dx, \qquad C(f) = \int_{-a}^{a} x^2 f(x)\,dx.$$

(a) What is $\sup\{A(f)^2 + B(f)^2 : f \in S_a\}$?

(b) What is $\sup\{A(f)^2 + B(f)^2 + C(f)^2 : f \in S_a\}$?

480. Let f be a twice differentiable function from $[0,1]$ to $\mathbb{R}$ with f'' continuous on $[0,1]$ and $\int_{1/3}^{2/3} f(x)\,dx = 0$. Prove

$$4860\left(\int_0^1 f(x)\,dx\right)^2 \leq 11\int_0^1 (f''(x))^2\,dx.$$

481. Let f be twice continuously differentiable on $[0,\pi]$ with $f(0)=f(\pi)=0$. Let $f(x)=\sum_{k=1}^{\infty} a_k \sin kx$ and $S_n(x)=\sum_{k=1}^{n} a_k \sin kx$. Show that, for every $n \in \mathbb{N}$,

$$\int_0^{\pi} (f(x)-S_n(x))^2\,dx \leq \frac{1}{3n^2}\int_0^{\pi} |f''(x)|^2\,dx.$$

482. Let f be a nonnegative, continuous, concave function on $[0,1]$ with $f(0)=1$. Prove that

$$2\int_0^1 x^2 f(x)\,dx + \frac{1}{12} \leq \left(\int_0^1 f(x)\,dx\right)^2.$$

(Seiffert's extension: for any $p>0$, $(p+1)\int_0^1 x^{2p} f(x)\,dx + \dfrac{2p-1}{8p+4} \leq \left(\int_0^1 f(x)\,dx\right)^2$.)

483. Let n be a natural number and let f be a continuous function from $[0,1]$ to $\mathbb{R}$ such that $\int_0^1 f^{2n+1}(x)\,dx = 0$. Prove that

$$\frac{(2n+1)^{2n+1}}{(2n)^{2n}}\left(\int_0^1 f(x)\,dx\right)^{4n} \leq \int_0^1 (f(x))^{4n}\,dx.$$

484. Let f be a real-valued function on $[0,1]$ such that f and its first two derivatives are continuous. Prove that if $f(1/2)=0$, then

$$\int_0^1 (f''(x))^2\,dx \geq 320\left(\int_0^1 f(x)\,dx\right)^2.$$

485. Let f be continuously differentiable on $[0,1]$, with $f(1/2)=0$ and $f^{(i)}(0)=0$ when i is even and less than n. Prove

$$\left(\int_0^1 f(x)\,dx\right)^2 \leq \frac{1}{(2n+1)4^n(n!)^2}\int_0^1 (f^{(n)}(x))^2\,dx.$$

486. Suppose $f : [0,1] \to \mathbb{R}$ is a differentiable function with continuous derivative and with $\int_0^1 f(x)\,dx = \int_0^1 xf(x)\,dx = 1$. Prove that

$$\int_0^1 |f'(x)|^3\,dx \geq \left(\frac{128}{3\pi}\right)^2.$$

487. Let k be a positive integer with $k \geq 2$, and let $f : [0,1] \to \mathbb{R}$ be a function with continuous k-th derivative. Suppose $f^{(k)}(x) \geq 0$ for all $x \in [0,1]$, and suppose $f^{(i)}(0) = 0$ for all $i \in \{0, 1, \ldots, k-2\}$. Prove

$$\int_0^1 x^{k-1} f(1-x)\,dx \leq \frac{(k-1)!\,k!}{(2k-1)!} \int_0^1 f(x)\,dx.$$

488. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f'(0)$ exists. Suppose $\alpha_n \neq \beta_n$ for all $n \in \mathbb{N}$ and both sequences $\{\alpha_n\}$ and $\{\beta_n\}$ converge to zero. Define $D_n = \dfrac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}$. Prove that $\lim_{n\to\infty} D_n = f'(0)$ under each of the following conditions:

(a) $\alpha_n < 0 < \beta_n$, $\forall n \in \mathbb{N}$.

(b) $0 < \alpha_n < \beta_n$ and $\dfrac{\beta_n}{\beta_n - \alpha_n} \leq M$, $\forall n \in \mathbb{N}$, for some $M > 0$.

(c) $f'(x)$ exists and is continuous for all $x \in (-1, 1)$.

489. Let α be a real number. Prove that there is no positive continuous function $f : [0,1] \to \mathbb{R}$ such that

$$\int_0^1 f(x)\,dx = 1, \qquad \int_0^1 xf(x)\,dx = \alpha, \qquad \int_0^1 x^2 f(x)\,dx = \alpha^2.$$

490. Let f be a continuous function on $[0,1]$. Prove that $(n+1)\displaystyle\int_0^1 x^n f(x)\,dx$ is in the range of f.

491. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $\forall x \in \mathbb{R}$, we have $f(x)^2 + f'(x)^2 \neq 0$. Show that for every bounded subset E of $\mathbb{R}$, f has at most a finite number of zeros in E.

492. Let f be a real continuous function with nonnegative values on the closed interval $[0,1]$. Set

$$u_n = \left(\int_0^1 (f(x))^n\,dx\right)^{1/n}, \qquad M = \sup_{0\le x\le 1} f(x).$$

(a) Assuming that $0<\varepsilon<1$, prove that there exist numbers α, β with $0\le\alpha<\beta\le 1$ such that $M(1-\varepsilon)\le f(x)\le M$ for all x in the open interval (α,β).

(b) Prove that $\lim_{n\to+\infty} u_n = M$.

Evaluate

$$D = \inf_{f\in\mathcal{F}}\left(\sup_{0\le t\le 1}|1-f(t)| + \int_0^1 |1-f(t)|\,dt\right),$$

where $\mathcal{F}$ is the vector space of all continuous functions from $[0,1]$ into $\mathbb{R}$ whose elements take the value zero at zero. Describe the geometric interpretation of the number D.

493. Let $f:[0,a]\to\mathbb{R}$ be a continuous and positive function. Prove that

$$\left(\int_0^a f(x)\,dx\right)\left(\int_0^a \frac{dx}{f(x)}\right)\ge a^2.$$

494. Let $f_1(x)$ be the inverse hyperbolic sine of x, and for $n\ge 2$, let $f_n(x) = f_1(f_{n-1}(x))$. For $b>0$, evaluate

$$\lim_{n\to\infty}\sqrt{n}\,f_n(b/\sqrt{n}).$$

495. Find

$$\int_0^{2026} f(x)\,dx,$$

where $f:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous function satisfying

$$f\left(\int_0^x t^t\,dt\right) = \int_0^x (f(t))^{f(t)}\,dt \qquad \text{for all } x\in\mathbb{R}^+.$$

496. Let f be a real-valued function on $[0,1]$ with continuous and non-negative fifth derivative such that

$$\int_0^1 f(x)\,dx = \int_0^1 x^5 f(x)\,dx = 0.$$

Prove that

$$\int_0^1 (56x^3+3x)f(x)\,dx \ge \int_0^1 (63x^4+21x^2)f(x)\,dx.$$

Find all the functions that give equality in the above inequality.

497. Let the sequence (u_n) be defined by $u_0 > 0$ and $u_{n+1} = \sqrt{u_n} + \frac{1}{n+2}$, $n = 0, 1, 2, \ldots$. Find $\lim u_n$.

498. Let (x_n) be defined by $x_1 = 2$, $x_2 = 3$ and for all $n \geq 2$,

$$x_{n+1} = \frac{n+1}{2n} x_n + \frac{n-2}{3n} x_{n-1} + \frac{1}{6}.$$

Prove that (x_n) has a finite limit and find it.

499. Let (u_n) satisfy $u_1 = 1$, $u_2 = \frac{2}{3}$, $u_{n+2} < \frac{1}{4} u_{n+1}^2 + \frac{3}{4} u_n$, $n = 1, 2, \ldots$. Prove that (u_n) has a limit and find it.

500. Let (x_n) be defined by $x_1 = 2$, $x_{n+1} = \frac{x_n^4 - x_n + 1}{x_n^3 - x_n + 1}$. Prove that $y_n = \sum_{k=1}^{n} \frac{1}{x_k^3}$ has a limit and compute it.

501. Let (x_n) be defined by $x_1 = 2$, $x_{n+1} = 1 + x_1 x_2 \cdots x_n$, for all $n \geq 1$. Compute $\lim \sum_{k=1}^{n} \frac{1}{x_k}$.

502. Let (x_n) be defined by $x_1 = 1$, $x_{n+1} = \sqrt{x_n(x_n+1)(x_n+2)(x_n+3)+1}$, for all $n \geq 1$. Compute $\lim \sum_{k=1}^{n} \frac{1}{x_k + 2}$.

503. Let (x_n) be defined by $x_1 = 5$, $x_{n+1} = x_n^2 - 2$, for all $n \geq 1$. Compute

$$\lim \frac{x_{n+1}}{x_1 x_2 \cdots x_n} \quad \text{and} \quad \lim \left(\frac{1}{x_1} + \frac{1}{x_1 x_2} + \cdots + \frac{1}{x_1 x_2 \cdots x_n} \right).$$

504.

1) Find the general term of (u_n) given that $u_1 = 2$, $u_2 = 8$, $u_{n+1} = 4u_n - u_{n-1}$ for all $n > 1$, $n \in \mathbb{N}$.

2) Using the result above, prove that the equation $\frac{x^2 + y^2}{xy + 1} = 4$ has infinitely many positive integer solutions.

505. Let (u_n) be defined by $u_1 = 2$, $u_{n+1} = \dfrac{u_n}{2u_n + 5}$, for all $n \in \mathbb{N}^*$.

1) Prove that (u_n) is decreasing.

2) Find the general formula for u_n.

506. Let n be a positive integer and let x_n be the unique positive root of the polynomial $f_n(t) = t^3 + 3t^2 - \dfrac{12}{n^2}$. Prove that the sequence (y_n) defined by $y_n = n(nx_n - 2)$ has a limit, and find it.

507. Let a be a real number and let (x_n) be defined by $x_1 = a$; $x_{n+1} = \dfrac{2017}{2018}\ln(2x_n^2 + 15) - 9$ for all $n \geq 1$. Prove that (x_n) has a finite limit as $n \to +\infty$.

508. Let (u_n) be defined by $u_1 = \dfrac{139}{2017}$, $u_n^2(2 - 9u_{n+1}) = 2u_{n+1}(2 - 5u_n)$, for all $n \geq 1$. Let

$$v_n = \frac{u_1}{1 - u_1} + \frac{u_2}{1 - u_2} + \cdots + \frac{u_n}{1 - u_n}.$$

Find $\lim v_n$.

509.

a) Let $q \in (0, 1)$ and let $\{u_n\}_{n\geq 1}$ satisfy $|u_{n+2} - u_{n+1}| < q|u_{n+1} - u_n|$ for all $n \geq 1$. Prove that $\{u_n\}$ has a finite limit.

b) Let $\{v_n\}_{n\geq 1}$ be defined by $0 < v_1 \neq 1$ and $v_{n+1} = \dfrac{3}{2 + v_n}$, for all $n \geq 1$. Prove that $\{v_n\}$ has a finite limit and find $\lim v_n$.

510. Consider the sequence (x_n) defined by $x_1 = \dfrac{5}{4}$ and $x_{n+1} = \dfrac{n + a}{n} + x_n - \dfrac{x_n^2}{2}$ for all $n = 1, 2, 3, \ldots$. Prove that (x_n) has a finite limit as $n \to +\infty$ and find it in the following cases:

1) $a = 0$.

2) $a = \dfrac{1}{2}$.

511. Let (x_n) be defined by $x_0 = 2017$; $x_n = -\dfrac{2017}{n}\displaystyle\sum_{k=0}^{n-1} x_k$ for $n \geq 1$. Find the limit

$$L = \lim \frac{n^2 \displaystyle\sum_{k=0}^{2017} 2^k x_k + 5}{-2018n^2 + 4n - 3}.$$

512. Find the general term of (u_n) given

$$u_1 = \frac{1}{2}, \quad u_2 = 673,$$
$$u_{n+2} = \frac{2(n+2)^2 u_{n+1} - (n^3 + 4n^2 + 5n + 2)u_n}{n+3}, \quad n \in \mathbb{N},\ n \geq 1.$$

513. Let $q \in (0, 1)$ be a real number. Consider two sequences $(a_n), (b_n)$ defined by $a_1 = 1$, $a_{n+1} = \sqrt{2 + a_n}$ and $b_n = a_n q^n + a_{n+1} q^{n-1} + \cdots + a_{2n}$ for every positive integer n. Prove that

1) $a_n \leq a_{n+1}$ for every positive integer n.

2) (b_n) has a finite limit equal to $\dfrac{2}{1-q}$.

514. Let (x_n) be defined by $x_1 = a > 0$; $x_{n+1} = x_n + \dfrac{n}{x_n}$, $n = 1, 2, 3, \ldots$.

1) Prove that $x_n \geq n$ for all $n \geq 2$.

2) Prove that $\left(\dfrac{x_n}{n}\right)$ has a finite limit and find it.

515. Let $(u_n)_{n=1}^{\infty}$ be defined by $u_1 = 1$; $u_{n+1} = \dfrac{4u_n + 3}{u_n + 2}$, for all $n \geq 1$. Prove that $(u_n)_{n=1}^{\infty}$ converges and find the limit.

516. Let (u_n) be defined by $u_1 > 0$, $u_{n+1} = \dfrac{2 + \sqrt{u_n^2 + 2u_n + 4}}{u_n}$, for all $n = 1, 2, 3, \ldots$. Prove that (u_n) has a finite limit and find it.

517. Let the real sequence (x_n) be defined by $x_0 = 1$, $x_1 = 5$, $x_{n+1} = 6x_n - x_{n-1}$ $(n \geq 1;\ n \in \mathbb{N})$. Find $\lim x_n\{\sqrt{2}\, x_n\}$, where $\{a\} = a - [a]$ denotes the fractional part of a.

518. Let (x_n) be defined by $x_0 = 1$, $x_1 = 5$, $x_{n+2} = \dfrac{x_{n+1} + \sqrt{x_n^2 + 6}}{3}$, for all $n \geq 0$. Prove that (x_n) has a finite limit and find it.

519. Let (u_n) be defined by $u_1 = 3$, $u_{n+1}^3 - 3u_{n+1}(u_{n+1} - 1) = u_n + 1$, for all $n \in \mathbb{N}^*$. Prove that (u_n) has a finite limit.

520. Let (x_n) be defined by $x_1 = 2017$, $x_{n+1} = \sqrt{3} + \dfrac{x_n}{\sqrt{x_n^2 - 1}}$, $(n \in \mathbb{N}^*)$. Prove that (x_n) has a limit and find it.

521. Let (x_n) be defined by $x_1 = 4$; $x_{n+1} = \dfrac{x_n^4 + 9}{x_n^3 - x_n + 6}$, for all $n \in \mathbb{N}^*$.

1) Prove that $\lim x_n = +\infty$.

2) For each positive integer n, let $y_n = \displaystyle\sum_{k=1}^{n} \frac{1}{x_k^3 + 3}$. Find $\lim y_n$.

522. Let (x_n), $n = 1, 2, 3, \ldots$, be defined by

$$x_1 = a \geq 1, \qquad x_{n+1} = \frac{x_n^2 - 2\{x_n\}^2}{[x_n]^2}; \quad n \in \mathbb{N}^*,$$

where $[x]$ is the integer part of x and $\{x\}$ is the fractional part of x.

1) For $a = 2017$, prove that (x_n) has a finite limit and find it.

2) Prove that (x_n) has a finite limit for all $a \geq 1$ and find it.

523. Let (x_n) be defined by $x_1 = 1$; $x_{n+1} = \dfrac{(2x_n - 1)^{10}}{52} + x_n$. For a positive integer n, let

$$u_n = \frac{(2x_1 - 1)^9}{2x_2 - 1} + \frac{(2x_2 - 1)^9}{2x_3 - 1} + \frac{(2x_3 - 1)^9}{2x_4 - 1} + \cdots + \frac{(2x_n - 1)^9}{2x_{n+1} - 1}.$$

Find $\lim u_n$.

524. Let (x_n) satisfy $x_1 = 3$, $x_2 = 7$, $x_{n+2} = x_{n+1}^2 - x_n^2 + x_n$ for all $n \in \mathbb{N}^*$. Let $y_n = \displaystyle\sum_{k=1}^{n} \frac{1}{x_k}$. Prove that (y_n) has a limit and find it.

525. Let (u_n) be defined by $u_1 = 2016$, $u_{n+1} = \dfrac{u_n^2}{1+u_n}$, for all $n \geq 1$.

1) Prove that (u_n) is decreasing.

2) Find the integer part of u_{1000}.

526. Let $a \geq 2$ be a real number and let (u_n) be defined by $u_1 = a$; $u_{n+1} = u_n + \ln \dfrac{u_n + 1}{2u_n - 3}$, $n = 1, 2, \ldots$. Prove that (u_n) has a finite limit, and find it.

527. Let the real sequence (x_n) be defined by $x_1 = 3$ and $x_{n+1} = \sqrt{21 + \sqrt{2x_n + 6}}$ for $n = 1, 2, \ldots$. Prove that (x_n) has a finite limit. Find it.

528. For each positive integer n, consider the function f_n on $\mathbb{R}$ defined by $f_n(x) = x^{2n} + x^{2n-1} + \cdots + x^2 + x + 1$.

1) Prove that f_n attains its minimum value at a unique point.

2) Let s_n be the minimum value of f_n. Prove that (s_n) has a finite limit.

529. Let (x_n) be defined by $x_0 > 0$; $x_{n+1} = \dfrac{x_n}{1+x_n^2}$, for all $n \in \mathbb{N}$. Find $\lim_{n\to+\infty} \sqrt{2n}\, x_n$.

530. Find all a for which (u_n) converges, given $u_1 = a$ and

$$u_{n+1} = \begin{cases} 2u_n - 1 & \text{if } u_n > 0 \\ -1 & \text{if } -1 \neq u_n \leq 0 \\ u_n^2 + 4u_n + 2 & \text{if } u_n < -1 \end{cases} \qquad \forall n \in \mathbb{N}.$$

531. Let a be a real number with $0 < a \leq 2$. Consider the sequence $(a_n)_{n\geq 0}$ defined by $a_0 = a$ and $a_{n+1} = \dfrac{1 + \sqrt{2a_n^2 + 1}}{a_n}$ for all $n \in \mathbb{N}$. Prove that $(a_n)_{n\geq 0}$ has a finite limit and find it.

532. Let a be a real number and let (x_n) be defined by $x_n = 2016n + a\sqrt[4]{n^3+1}$, for all $n = 1, 2, \ldots$.

1) Find a so that (x_n) has a finite limit.

2) Find a so that (x_n) is eventually increasing.

533. Let a, b be positive integers. Consider (u_n) defined by $u_n = a^2n^2 + bn$, for $n = 1, 2, \ldots$. Compute $\lim_{n\to\infty}\{\sqrt{u_n}\}$, where $\{x\}$ denotes the fractional part of the real number x.

534. Consider (u_n) defined by $u_1 = 1$, $u_2 = \dfrac{3}{2}$, $u_{n+2} = \dfrac{u_{n+1}+2}{u_n+2}$, $n \geq 1$. Prove that (u_n) has a finite limit and find it.

535. Let (a_n) have $a_1 \in \mathbb{R}$ and $a_{n+1} = |a_n - 2^{1-n}|$ for all $n \in \mathbb{N}^*$. Find $\lim a_n$.

536. For $n \in \mathbb{N}^*$, let A_n be the set of numbers with decimal representation $0.a_1a_2\ldots a_n$, where $a_n = 1$ and each $a_i \in \{0, 1\}$ for $i = 1, 2, \ldots, n-1$. Let x_n be the number of elements of A_n and y_n the sum of the elements of A_n.

a) Find a formula for x_n in terms of n.

b) Compute $\lim_{n\to+\infty} \dfrac{y_n}{x_n}$.

537. For $f \in C^3([0,1])$ with $f'(0) = f'(1) = 0$, $f''' > 0$ on $(0,1)$, prove there is exactly one $c \in (0,1)$ with $f(c) - f(0) = cf'(c)$ and $f''(c) > 0$.

538. For monotone decreasing $f_n : [0,1] \to [0,1]$ and A_n defined recursively by $A_1 = \int_0^1 f_1$, $A_n = \int_0^{A_{n-1}} f_n$, prove $\int_0^1 f_1f_2\cdots f_{2024} \leq A_{2024}$.

539. Let $f : [0,1] \to [0,\infty)$ be a differentiable function such that $f'(x)$ is decreasing, $f(0) = 0$, and $f'(1) > 0$. Prove that the following inequality holds:

$$\int_0^1 \frac{1}{f(x)^2+1}\,dx \leq \frac{f(1)}{f'(1)}.$$

Can equality be achieved?

540. Let $f \in L^2(0,\infty)$ and $f'' \in L^2(0,\infty)$. Prove that $f' \in L^2(0,\infty)$ and

$$\left(\int_0^\infty |f'(x)|^2\,dx\right)^2 \leq 4\int_0^\infty |f(x)|^2\,dx \int_0^\infty |f''(x)|^2\,dx.$$

541. Let $f : [0,1] \to \mathbb{R}$ be a continuous function satisfying

$$\int_0^1 x^2 f(x)\,dx = 0, \qquad \int_0^1 f(x)\,dx = 0.$$

Find the smallest constant λ such that

$$\left|\int_0^1 xf(x)\,dx\right| \le \lambda \max_{x\in[0,1]} |f(x)|.$$

542. Let n be a positive integer and let $f:[0,1]\to\mathbb{R}$ be a continuous function such that

$$\int_0^1 x^k f(x)\,dx = 1 \quad \text{for every } k\in\{0,1,\ldots,n-1\}.$$

Prove that

$$\int_0^1 (f(x))^2\,dx \ge n^2.$$

543. Let $f(x)$ be a differentiable function on $[0,1]$ with $f(0)=0$ and $f(1)=1$. Show that for arbitrary positive real numbers a_1,a_2 there always exist distinct real numbers $x_1,x_2\in[0,1]$ such that

$$a_1+a_2=\frac{a_1}{f'(x_1)}+\frac{a_2}{f'(x_2)}.$$

544. Calculate

$$\lim_{x\to 0}\left(\frac{1}{x}\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{x}{n}\right)^n\right)^{1/x}.$$

545. Let α be a positive real number and f be a nonnegative function on $[0,1]$ such that

$$\int_x^1 (f(t))^\alpha\,dt \ge \int_x^1 t^\alpha\,dt \quad \text{for all } 0\le x\le 1.$$

Prove that

$$\int_0^1 (f(t))^{\alpha+\beta}\,dt \ge \int_0^1 (f(t))^\alpha t^\beta\,dt \ge \int_0^1 t^{\alpha+\beta}\,dt$$

for every positive real β.

546. Let $f:[a,b]\to\mathbb{R}$ be an n times continuously differentiable function on (a,b), and let $a_1,a_2,\ldots,a_{n+1}$ be $n+1$ distinct numbers in (a,b). Prove that there exists c in (a,b) such that

$$\sum_{j=1}^{n+1} f(a_j)\prod_{\substack{1\le i\le n+1\\ i\ne j}}(a_j-a_i)^{-1}=\frac{f^{(n)}(c)}{n!}.$$

547. Let $f : [0,1] \to \mathbb{R}$ be a differentiable and continuous function such that $f(0) = 0$ and $f(1) = 1$. Prove that

$$\int_0^1 |f'(x) - xf(x)|\, dx > \frac{1}{\sqrt{e}}.$$

548. A function f is differentiable on $[0,2]$ and $f(0) = 0$, $f(1) = 2$, $f(2) = 1$. Prove that $f'(c) = 0$ for some c in $(0,2)$.

549. Let $f : [0,1] \to \mathbb{R}$ be a continuous function on $[0,1]$.

(a) Prove that there exists a point $c \in (0,1)$ such that

$$\int_0^1 f\, dx = f(c).$$

(b) Prove that there exist two distinct points $a, b \in (0,1)$ such that

$$\left(\int_0^1 f\, dx\right)^2 = f(a)f(b).$$

(c) Given a natural number $n \geq 3$, does there exist n distinct points $a_i \in (0,1)$, $1 \leq i \leq n$, such that

$$\left(\int_0^1 f\, dx\right)^n = f(a_1)f(a_2)\cdots f(a_n)?$$

(If the answer is "yes," prove it; if the answer is "no," give a counterexample for some value of n for which the claim fails.)

550. Let $f : [0,1] \to [0,1]$ be a continuous function (not necessarily differentiable) such that $f(f(x)) = 1$ for all $x \in [0,1]$. Prove that

$$\int_0^1 f(x)\, dx > \frac{3}{4}.$$

551. Let $f : [0,1] \to \mathbb{R}$ be continuously differentiable with $f(0) = 0$ and $0 < f'(x) \leq 1$. Prove that

$$\left(\int_0^1 f(x)\, dx\right)^2 \geq \int_0^1 [f(x)]^3\, dx.$$

552. Let (u_n) be defined by $u_1 = 3$, $u_2 = 7$, and

$$\frac{u_{n+1} - u_n}{u_n - u_{n-1}} = u_n + u_{n-1}, \qquad \forall n \geq 2.$$

Compute

$$L = \lim_{n \to +\infty} \left(\frac{1}{u_1} + \frac{1}{u_1 u_2} + \frac{1}{u_1 u_2 u_3} + \cdots + \frac{1}{u_1 u_2 \cdots u_n} \right).$$

553. Let (x_n) satisfy $x_1 = 1$ and

$$x_{n+1} = \frac{1 + \sqrt{1 + x_n^2} - x_n}{1 + \sqrt{1 + x_n^2} + x_n}, \qquad \forall n \in \mathbb{N}^*.$$

Prove that (x_n) has a finite limit and find it.

554. Let a be a real number and (u_n) be defined by

$$u_1 = \frac{3}{4}, \quad u_2 = \frac{1}{2}, \quad u_{n+2} = \frac{u_{n+1}^3 + 2u_n^3 + a}{4}, \quad n = 1, 2, \ldots$$

Find the largest value of a for which (u_n) has a finite limit, and in that case find $\lim u_n$.

555. Let (u_n) satisfy $u_1 = 16$ and

$$u_{n+1} + 14 = \frac{15(nu_n + 1)}{n + 1}, \qquad \forall n \in \mathbb{N}^*.$$

Compute

$$L = \lim_{n \to +\infty} \frac{n(u_n - 1) + 14^n}{2024 \cdot 15^n}.$$

556. Let (u_n) satisfy $u_1 = 2026$ and

$$u_{n+1} = \frac{u_n(u_n^2 + 3)}{3u_n^2 + 1}, \qquad \forall n \geq 1.$$

Prove that (u_n) has a finite limit and find it.

557. Let (a_n) satisfy $|a_{m+n} - a_m - a_n| \leq 1$ for all $m, n \in \mathbb{N}^*$. Prove that $\left(\frac{a_n}{n}\right)$ converges.

558. Let (x_n) satisfy

$$x_1 = 1, \quad x_{n+1} - \sqrt{x_{n+1}} = x_n + \sqrt{x_n} + \frac{1}{n+4}, \qquad n = 1, 2, \ldots$$

Prove that $y_n = \dfrac{x_n}{n^2}$ has a limit, and find it.

559. Let $f(n) = (n^2 + n + 1)^2 + 1$. Consider the sequence

$$u_n = \frac{f(1)f(3)\cdots f(2n-1)}{f(2)f(4)\cdots f(2n)}.$$

Compute $\lim n\sqrt[n]{u_n}$.

560. Let (u_n) be an infinite decreasing geometric sequence with positive ratio satisfying $u_6 = \dfrac{1}{8}$ and $\dfrac{u_5 - u_1}{u_3 - u_1} = \dfrac{5}{4}$. Let (v_n) be defined by $v_n = u_n^2$. Compute the sum of all terms of (v_n).

561. Let $a \in (0, 1)$ be a real number, and let (u_n) be defined by $u_1 = 1$ and

$$u_{n+1} = \sqrt[3]{au_n^3 + a - 1}, \qquad \forall n \in \mathbb{N}^*.$$

(a) Let (v_n) be defined by $v_n = u_n^3 + 1$. Prove that (v_n) is an infinite decreasing geometric sequence.

(b) Find all values of a for which $\lim \left(u_1^3 + u_2^3 + \cdots + u_n^3 + n\right) = 4$.

562. Let (u_n) satisfy $u_1 = 0$ and $u_{n+1} = u_n + 4n + 3$, $\forall n \geq 1$. Compute

$$\lim \frac{\sqrt{u_n} + \sqrt{u_{4n}} + \sqrt{u_{4^2 n}} + \cdots + \sqrt{u_{4^{2026} n}}}{\sqrt{u_n} + \sqrt{u_{2n}} + \sqrt{u_{2^2 n}} + \cdots + \sqrt{u_{2^{2026} n}}}.$$

563. Let (u_n) satisfy $u_1 = 5$, $u_{n+1} = u_n^{2026} - 3u_n^{2025} + u_n$.

(a) Prove that $\lim\limits_{n \to +\infty} u_n = +\infty$.

(b) For each positive integer n, let

$$v_n = \frac{1}{(u_1^{2025} + 1)(u_2^{2025} + 1)\cdots(u_n^{2025} + 1)}.$$

Compute $\lim\limits_{n \to +\infty} v_n$.

564. Let (u_n) satisfy $u_1 = 2026$, $u_{n+1} = u_n^{2027} + 2026u_n^{2026} + u_n$.

(a) Prove that $\lim\limits_{n\to+\infty} u_n = +\infty$.

(b) Compute

$$\lim \left(\frac{u_1^{2026}}{u_2 + 2026} + \frac{u_2^{2026}}{u_3 + 2026} + \cdots + \frac{u_n^{2026}}{u_{n+1} + 2026} \right).$$

565. Let (u_n) be defined by $u_1 = -2026$, $u_{n+1} = u_n - 2026$. Compute

$$\lim \left(\frac{1}{u_1\sqrt{-u_2} + u_2\sqrt{-u_1}} + \frac{1}{u_2\sqrt{-u_3} + u_3\sqrt{-u_2}} \right.$$
$$\left. + \cdots + \frac{1}{u_n\sqrt{-u_{n+1}} + u_{n+1}\sqrt{-u_n}} \right).$$

566. Let $(a_n), (b_n)$ be defined by

$$a_1 = \frac{2015 + 2016}{2},\ b_1 = \sqrt{2016a_1},$$
$$a_2 = \frac{a_1 + b_1}{2},\ b_2 = \sqrt{a_2 b_1},\ \ldots$$
$$a_n = \frac{a_{n-1} + b_{n-1}}{2}, \qquad b_n = \sqrt{a_n b_{n-1}}.$$

Compute $\lim b_n$ and $\lim a_n$.

567. Let (u_n) be defined by $u_1 = 6$ and

$$u_{n+1} = \frac{u_n^2 + n(u_n - 1) + n^2}{n}, \qquad \forall n \geq 1.$$

Compute

$$\lim \left(\frac{1}{u_1} + \frac{1}{u_2} + \cdots + \frac{1}{u_n} \right).$$

568. Let (u_n) be defined by $u_1 = 1$ and

$$u_{n+1} = \frac{u_n}{\sqrt{nu_n^2 + 1}}, \qquad \forall n \in \mathbb{N}^*.$$

(a) Find the general term of (u_n).

(b) Let $[x]$ denote the integer part of x and $\{x\} = x - [x]$ its fractional part. For each positive integer n, let $x_n = nu_n$. Compute $\lim\limits_{n\to+\infty} \{x_n\}$.

569. Let (u_n) be a geometric sequence with first term $u_1 = 1$ and common ratio $q = 3$. Prove that

$$\frac{1}{u_3 - u_1} + \frac{1}{u_4 - u_2} + \cdots + \frac{1}{u_{2026} - u_{2024}} < \frac{2}{3}.$$

570. Let (a_n) be defined by $a_1 = 3$, $a_2 = 7$, $a_{n+2} = 3a_{n+1} - a_n$, $n = 1, 2, \ldots$

(a) Prove that $\dfrac{a_1^2}{7} + \dfrac{a_2^2}{7^2} + \cdots + \dfrac{a_n^2}{7^n} < \dfrac{142}{3}$, $\forall n \in \mathbb{N}^*$.

(b) Let $b_n = \dfrac{1}{a_1 a_2} + \dfrac{1}{a_2 a_3} + \cdots + \dfrac{1}{a_n a_{n+1}}$, $\forall n \in \mathbb{N}^*$. Prove that (b_n) has a finite limit and find it.

571. Suppose $0 < a < \pi/2$. Let (u_n), $n = 1, 2, 3, \ldots$, be defined by $u_1 > 0$ and

$$u_{n+1} = u_n \sin^2 a + \frac{2002 \cos^2 a}{u_n \tan^2 a}, \qquad n = 1, 2, 3, \ldots$$

Prove that (u_n) has a limit as $n \to +\infty$ and find it.

572. Let $f_n(x)$, $n = 0, 1, 2, \ldots$, be a sequence of functions satisfying:

(i) $f_n(x) : (0, +\infty) \to (0, +\infty)$, $n = 0, 1, 2, \ldots$

(ii) $f_0(x) = x$, $\forall x \in (0, +\infty)$

(iii) $f_{n+1}(x) = \sqrt{x^2 + 6f_n(x)}$, $\forall x \in (0, +\infty)$, $\forall n \geq 0$.

(a) Prove that for each $n = 1, 2, \ldots$ there exists a unique $x_n > 0$ satisfying $f_n(x_n) = 2x_n$.

(b) Prove that (x_n) has a finite limit and find it.

573. Prove that there is no function $\varphi(t)$ satisfying both of the following conditions:

- $\varphi(t)$ is bounded on $[0, 2]$;
- $\varphi(t + h) \geq 1 + 2h\varphi^2(t)$ for all $t, h \geq 0$ with $t + h \leq 2$.

574. Find all functions $f(x)$ defined and twice differentiable on $(0, +\infty)$ satisfying $f'(x) > 0$ and $f(f'(x)) = -f(x)$ for all $x \in (0, +\infty)$.

575. Let $f(x)$ be twice continuously differentiable on $(0,\infty)$ and satisfy $\lim_{x\to\infty} xf(x) = 0$ and $\lim_{x\to\infty} xf''(x) = 0$. Compute $\lim_{x\to\infty} xf'(x)$.

576. Find all continuously differentiable functions $f(x)$ on $\mathbb{R}$ satisfying $|f(x)| < 2$ and $f(x)f'(x) \geq \sin x$ for all $x \in \mathbb{R}$.

577. Let $\{x_n\}$ be defined by $x_1 = 1$, $x_n = a^{x_{n-1}}$ $(a > 0)$. Find the largest value of a for which $\{x_n\}$ has a limit.

578. Let $\{x_n\}$ be bounded and satisfy $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$. Can we conclude that $\{x_n\}$ converges?

579. Let $f, g : [0,1] \to [0,1]$ be continuous, with f increasing. Prove that

$$\int_0^1 f(g(x))\,dx \leq \int_0^1 (f(x) + g(x))\,dx.$$

580. Let $x_1 = a \in (0,1)$ and $x_{n+1} = \ln(x_n + 1)$. Find $\lim_{n\to+\infty} nx_n$.

581. Let $f(x)$ be twice differentiable on $[a,b]$ with $f''(x) \geq 0$ and $f(x) > 0$ on $[a,b]$. Prove that

$$\frac{1}{b-a}\int_a^b f(x)\,dx \geq \frac{1}{2} f\left(\frac{a+b}{2}\right).$$

582. Let $f(x)$ be defined and increasing on all of $\mathbb{R}$. Suppose $f(x) - x$ is periodic and there exists x_0 with $f(x_0) = x_0$. Let

$$f_1(x) = f(x),\ f_2(x) = f(f(x)),\ f_3(x) = f(f_2(x)),\ \ldots,\ f_n(x) = f(f_{n-1}(x)).$$

Compute $\lim_{n\to\infty} \dfrac{f_n(x)}{n}$.

583. Let $P(x) \in \mathbb{R}_n[x]$ be a polynomial of degree $n \geq 1$ satisfying $P(x) \geq 0$ for all $x \in \mathbb{R}$. Prove that for all $x \in \mathbb{R}$,

$$\sum_{j=0}^{n} P^{(j)}(x) \geq 0.$$

584. Let V be the class of functions $f(x)$ continuous on $[0,1]$, differentiable on $(0,1)$, satisfying $f(0)=0$, $f(1)=1$. Find all constants $\alpha \in \mathbb{R}$ with the property that for every $f \in V$ there exists $\xi \in (0,1)$ such that

$$f'(\xi) = 2025 f(\xi) + \alpha.$$

585. Let $f(x)$ be continuously differentiable on $[a,b]$ satisfying $f(a)=f(b)=0$. Prove that

$$\max_{x\in[a,b]} |f'(x)| \geq \frac{1}{(b-a)^2}\int_a^b |f(x)|\,dx.$$

586. Find all functions $f(x)$ satisfying $\dfrac{f(1)}{f(0)} = e$ and

$$\int_0^1 \frac{dx}{(f(x))^2} + \int_0^1 (f'(x))^2\,dx \leq 2.$$

587. Let $f:[a,b]\to\mathbb{R}$ be a differentiable function satisfying $f(a)=0$ and $0\leq f'(x)\leq 1$ for all $x\in[a,b]$. Prove that

$$\frac{4}{9}f^3(x) \leq \left(\int_a^x \sqrt{f(t)}\,dt\right)^2, \qquad a\leq x\leq b,$$

and

$$4\int_a^b \sqrt{f^7(x)}\,dx \leq 3\left(\int_a^b \sqrt{f(x)}\,dx\right)^3.$$

588. Let f be a continuous function on $[0,1]$ such that $f(0)=0$. Prove that

$$\lim_{a\to 0^+}\int_0^1 a x^{a-1} f(x)\,dx = 0.$$

589. Let $n\in\mathbb{Z}^+$ such that $n\leq e^{4047}$. Let $(f_1(x), f_2(x), \ldots, f_n(x))$ and $(g_1(x), g_2(x), \ldots, g_n(x))$ be permutations of $(x, x^2, \ldots, x^n)$. Is it possible that

$$\int_0^1 \left[x f_1(x) + x^2 f_2(x) + \cdots + x^n f_n(x)\right] dx$$
$$> 2024\int_0^1 \left[x g_1(x) + x^2 g_2(x) + \cdots + x^n g_n(x)\right] dx\ ?$$

590. If $f:[a,b]\to(0,\infty)$, where $0<a<b$, is a continuous increasing function, then

$$\left(\int_a^b xf(x)\,dx\right)\left(\int_a^b f^2(x)\,dx\right)\left(\int_a^b x^3f(x)\,dx\right)\geq\frac{a^2b^2}{b-a}\left(\int_a^b f(x)\,dx\right)^4.$$

591. Let $f,g:[0,1]\to(0,\infty)$ be two distinct continuous functions correlated by the equality $\int_0^1 f(x)\,dx=\int_0^1 g(x)\,dx$, and define the sequence $(x_n)_{n\geq 0}$ as

$$x_n=\int_0^1\frac{(f(x))^{n+1}}{(g(x))^n}\,dx.$$

(a) Show that $\lim\limits_{n\to\infty} x_n=\infty$.

(b) Demonstrate that the sequence $(x_n)_{n\geq 0}$ is monotone.

592. Find all pairs of twice differentiable functions $f,g:\mathbb{R}\to\mathbb{R}$, with their second derivative being continuous, such that the following holds for all $x,y\in\mathbb{R}$:

$$(f(x)-g(y))(f'(x)-g'(y))(f''(x)-g''(y))=0.$$

593. Let a real-valued sequence $\{x_n\}_{n\geq 1}$ be such that

$$\lim_{n\to\infty} nx_n=0.$$

Find all possible real values of t such that $\lim\limits_{n\to\infty} x_n(\log n)^t=0$.

594. For a given integer $n\geq 1$, let $f:[0,1]\to\mathbb{R}$ be a non-decreasing function. Prove that

$$\int_0^1 f(x)\,dx\leq(n+1)\int_0^1 x^n f(x)\,dx.$$

Find all non-decreasing continuous functions for which equality holds.

595. We define the sequence $(x_n)_{n\in\mathbb{N}}$ by $x_0=a$, $x_1=b$ and

$$x_{n+1}=\frac{1}{2}\left(x_{n-1}+\frac{1}{x_n}\right).$$

Show that if the sequence (x_n) is periodic (i.e., $\exists T\in\mathbb{N}^*$, $x_{n+T}=x_n$), then $ab=1$.

596. Let $f:[0,1]\to\mathbb{R}$ be a continuous function satisfying $\int_0^1 f(x)\,dx=1$. Evaluate

$$\lim_{n\to\infty}\frac{n}{\ln n}\int_0^1 x^n f(x^n)\ln(1-x)\,dx.$$

597. Let f be a real-valued function on $[0,1]$ with a continuous second derivative. Assume that $f(0)=0$, $f'(0)=1$, $f''(0)\neq 0$ and $0<f'(x)<1$ for all $x\in(0,1]$. Let $x_1,x_2,\ldots$ be a sequence with $0<x_1\le 1$ and with

$$x_n=f\left(\frac{x_1+x_2+\cdots+x_{n-1}}{n-1}\right)\quad\text{for } n\ge 2.$$

Prove that

$$\lim_{n\to\infty}x_n\ln n=-\frac{2}{f''(0)}.$$

598. Suppose that $f:[0,1]\to\mathbb{R}$ has a continuous and nonnegative third derivative, and suppose $\int_0^1 f(x)\,dx=0$. Prove

$$10\int_0^1 x^3 f(x)\,dx+6\int_0^1 xf(x)\,dx\ge 15\int_0^1 x^2 f(x)\,dx.$$

599. If $n\in\mathbb{N}$ and $n\ge 2$, also $f:[1,n]\to(0,\infty)$ and f is integrable and $i\in\{1,\ldots,n-1\}$, and let

$$\Omega=\int_1^n f(x)\,dx,\qquad 0\le\alpha\le\min_i\left(\int_i^{i+1}f(x)\,dx\right)\le\max_i\left(\int_i^{i+1}f(x)\,dx\right)\le\beta,$$

then prove that

$$(n-1)\alpha\beta+\sum_{i=1}^{n-1}\left(\int_i^{i+1}f(x)\,dx\right)^2\le(\alpha+\beta)\Omega.$$

600. If $a>0$, $f:[0,a]\to\mathbb{R}$, $f'(a)=f(a)=0$ and $f\in C^2([0,a])$, $f',f''\in C^1([0,a])$, then prove that

$$60\left(\int_0^a f(x)\,dx\right)^4\le a^8\left(\int_0^a (f'(x))^2\,dx\right)\left(\int_0^a (f''(x))^2\,dx\right).$$

601. Let $a_0\ge 0$, and define a sequence by $a_{n+1}=a_0a_1\cdots a_n+4$ for all $n\ge 0$. Prove that for every $n\ge 1$,

$$a_n-\sqrt[4]{(a_{n+1}+1)(a_n^2+1)-4}=1.$$

602. A sequence $(a_n)_{n\geq 0}$ of real numbers satisfies $a_0 = 1$ and

$$a_{n+1} = \frac{a_n}{n^2 a_n + a_n^2 + 1}, \qquad n \geq 0.$$

Find $\lim_{n\to\infty} n^3 a_n$.

603. Let $(a_n)_{n\geq 1}$ be an increasing sequence of positive real numbers with $\lim_{n\to\infty} a_n = \infty$, and suppose the sequence of consecutive differences $(a_{n+1} - a_n)_{n\geq 1}$ is monotone. Compute

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n\sqrt{a_n}}.$$

604. Let a, b, c be positive real numbers with $b^2 > 4ac$. Suppose a sequence $(\lambda_n)_{n\geq 0}$ satisfies $\lambda_0 > 0$ and $c\lambda_1 > b\lambda_0$, and define a related sequence $(u_n)_{n\geq 0}$ by

$$u_0 = c\lambda_0, \qquad u_1 = c\lambda_1 - b\lambda_0, \qquad u_n = a\lambda_{n-2} - b\lambda_{n-1} + c\lambda_n \quad (n \geq 2).$$

Prove that if $u_n > 0$ for every $n \geq 0$, then $\lambda_n > 0$ for every $n \geq 0$.

605. Let $0 \leq a \leq 2$, and define a sequence $(a_n)_{n\geq 1}$ by $a_1 = a$ and

$$a_{n+1} = 2^n - \sqrt{2^n(2^n - a_n)}, \qquad n \geq 1.$$

Compute $\sum_{n=1}^{\infty} a_n^2$.

606. Let $(x_n)_{n\geq 1}$ be a monotone sequence of real numbers, and let $a \in (-1, 0)$. Find

$$\lim_{n\to\infty} \left(x_1 a^{n-1} + x_2 a^{n-2} + \cdots + x_{n-1} a + x_n\right).$$

607. Compute $\lim_{n\to\infty} a_n$ for the sequence defined by $a_1 = 1$, $a_2 = 2$, and for every positive integer k,

$$a_{2k+1} = \frac{a_{2k-1} + a_{2k}}{2}, \qquad a_{2k+2} = \sqrt{a_{2k}\, a_{2k+1}}.$$

608. Let p, q be real numbers with $p > 0$ and $q > -p^2/4$. Define $U_0 = 0$, $U_1 = 1$, and $U_{n+2} = pU_{n+1} + qU_n$ for $n \geq 0$. Compute

$$\lim_{n\to\infty} \sqrt{U_1^2 + \sqrt{U_2^2 + \sqrt{U_4^2 + \sqrt{\cdots + \sqrt{U_{2^{n-1}}^2}}}}}\,.$$

609. Let $a > 0$ be a real number, and define a sequence $(x_n)_{n\geq 1}$ by

$$x_1 = 1, \qquad x_{n+1} = x_n + an \prod_{i=1}^{n} x_i^{-1/n} \quad (n \geq 1).$$

(1) Prove that $\lim\limits_{n\to\infty} x_n = \infty$.

(2) Find $\lim\limits_{n\to\infty} \dfrac{x_n}{\ln n}$.

610. Let $a > 0$ be a real number, and define a sequence $(x_n)_{n\geq 1}$ by $x_1 = 1$ and

$$x_{n+1} = x_n + \frac{a}{x_1 + x_2 + \cdots + x_n}, \qquad n \geq 1.$$

(1) Prove that $\lim\limits_{n\to\infty} x_n = \infty$.

(2) Find $\lim\limits_{n\to\infty} \dfrac{x_n}{\sqrt{\ln n}}$.

611. Let $(x_n)_{n\geq 0} \subset [0,1]$ satisfy $x_0 = 1$ and, for a fixed real number $\alpha > 1$,

$$(1 - x_n)^2 - (1 - x_{n-1})^2 = \frac{x_n + x_{n-1}}{\alpha}, \qquad n \geq 1.$$

Prove that

$$\sum_{n=1}^{\infty} x_n = \frac{\alpha - 1}{2}.$$

612. Let a_1 be a positive real number, and define a sequence by $a_{n+1} = n^2/a_n$ for $n = 1, 2, \ldots$. Compute

$$\lim_{n\to\infty} \frac{1}{\ln n} \sum_{k=1}^{n} \frac{1}{a_k}.$$

613. Let a, b, c be positive real numbers, each at most 2. Define $(x_n)_{n\geq 0}$ by $x_0 = a$, $x_1 = b$, $x_2 = c$, and

$$x_{n+1} = \sqrt{x_n + \sqrt{x_{n-1} + x_{n-2}}}, \qquad n \geq 2.$$

Prove that $(x_n)_{n\geq 0}$ converges, and determine its limit.

614. Let $\{a_n\}_{n\in\mathbb{Z}}$ be a two-sided sequence with $a_0 = 1$, $a_1 = 0$, $a_2 = 0$, and

$$a_{n+3} = a_n + a_{n+1} + a_{n+2}$$

for every integer n. Prove that

$$a_n\, a_{-n} = a_{n+1}\, a_{-n+1} + a_{n+2}\, a_{-n+2} + a_{n+3}\, a_{-n+3}.$$

615. Define a sequence of positive reals $(a_n)_{n\geq 1}$ by $a_1 = 3$ and $a_n = \frac{1}{2}\left(a_{n-1}^2 + 1\right)$ for $n > 1$. Prove that

$$\left[\left(\sum_{k=1}^{n} \frac{a_k}{1+a_k}\right)\left(\sum_{k=1}^{n} \frac{1}{a_k(1+a_k)}\right)\right]^{1/2} \leq \frac{1}{4}\cdot\frac{a_1 + a_n}{\sqrt{a_1 a_n}}.$$

616. Let q be a positive real number with $q \neq 1$, and define

$$v_n = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}, \qquad \mu_n = q^{n/2} + q^{-n/2}.$$

(a) Prove that $2v_{n+1} = \mu_1 v_n + \mu_n$ for every $n \geq 1$.

(b) Prove that $2v_{n-1} = \mu_1 v_n - \mu_n$ for every $n \geq 2$.

(c) Find a closed form for $\displaystyle\sum_{n=2}^{N} \frac{1}{v_n\, v_{n+1}}$.

617. Fix positive real numbers x_1, x_2, and for $n \geq 2$ define a sequence by

$$x_{n+1} = \sqrt[n]{x_1} + \sqrt[n]{x_2} + \cdots + \sqrt[n]{x_n}.$$

Find $\displaystyle\lim_{n\to\infty} \frac{x_n - n}{\ln n}$.

618. Prove that for every integer $n \geq 4$ there exist integers $x_1, x_2, \ldots, x_n$ satisfying

$$\frac{x_{n-1}^2 + 1}{x_n^2} \prod_{k=1}^{n-2} \frac{x_k^2 + 1}{x_k^2} = 1,$$

together with all of the following conditions:

- $x_1 = 1$, and $x_{k-1} < x_k < 3x_{k-1}$ for every $2 \leq k \leq n-2$;
- $x_{n-2} < x_{n-1} < 2x_{n-2}$ (for $n \geq 3$);
- $x_{n-1} < x_n < 2x_{n-1}$ (for $n \geq 2$).

619.

1. Does there exist a continuous and bounded function

   $$f : [0, +\infty) \to \mathbb{R}$$

   such that, for every $x \geq 0$,

   $$f(x) = 1 + \int_0^x \frac{e^{-t^2}}{1 + f(t)^2}\, dt?$$

2. Study the sequence of functions

   $$f_n : [0, +\infty) \to \mathbb{R}$$

   defined by

   $$f_0(x) = 0,$$

   and, for every $n \in \mathbb{N}$ and $x \geq 0$,

   $$f_{n+1}(x) = 1 + \int_0^x \frac{e^{-t^2}}{1 + f_n(t)^2}\, dt.$$

620. Let $f : [0,1] \to \mathbb{R}$ be continuous, and let α, β be fixed real numbers. Define

$$x_n = \sum_{k=0}^{n} \cos\left(\frac{1}{\sqrt{n}} f\left(\frac{k}{n}\right)\right) - \alpha n^{\beta}, \qquad n \geq 1.$$

Compute $\lim_{n\to\infty} x_n$, by evaluating all cases according to the values of α and β.

621. Let $p > 1$ be an integer. Prove that

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{\sqrt[p]{k}}-\frac{p}{p-1}\left(n^{(p-1)/p}-1\right)\right)$$

exists and lies strictly between 0 and 1.

622. Let (u_n) be defined by $u_0 \in (-1,1)$ and $u_{n+1} = \sqrt{\dfrac{1+u_n}{2}}$ for all $n \in \mathbb{N}$. Let $v_n = 4^n(1-u_n)$. Prove that (v_n) has a finite limit and find it.

623. Let (a_n) be defined by $a_1 = \dfrac{1}{2}$ and

$$a_{n+1} = \frac{1}{2}\left[\sqrt{a_n^2+\frac{1}{4^n}}+a_n\right], \quad n = 1, 2, \ldots$$

Prove that (a_n) converges and find $\lim\limits_{n\to+\infty} a_n$.

624. Let (x_n) satisfy $x_0 = 3$ and $x_{n+1}^3 - 3x_{n+1} = \sqrt{x_n+2}$ for all $n \in \mathbb{N}$. Prove that the sequence has a finite limit and find it.

625. Let (x_n) be defined by $x_1 = 1$ and $x_n = \dfrac{2n}{(n-1)^2}\displaystyle\sum_{i=1}^{n-1} x_i$ for $n \geq 2$. Let $y_n = x_{n+1} - x_n$, $n = 1, 2, 3, \ldots$. Prove that the sequence (x_n) has a finite limit as $n \to +\infty$.

626. Let (x_n) be defined by $x_0 = -2$ and $x_{n+1} = \dfrac{1-\sqrt{1-4x_n}}{2}$ for all $n \in \mathbb{N}$. Let $u_n = (1+x_0^2)(1+x_1^2)\cdots(1+x_n^2)$. Prove that (u_n) has a finite limit.

627. Let (a_n) satisfy $a_1 = \dfrac{1}{2}$, where for $n \geq 1$, a_{n+1} is the unique real number satisfying $a_n = 3a_{n+1} - a_{n+1}^3$ with $0 \leq a_n \leq 1$ for all $n \geq 1$.

(1) Prove that the sequence (a_n) is well-defined and has a finite limit.

(2) Let (b_n) be defined by $b_n = (1+2a_1)\left(1+2^2a_2\right)\cdots(1+2^na_n)$ for all $n \geq 1$. Prove that (b_n) has a finite limit.

628. Let (a_n) be defined by $a_1 = 1$ and $a_{n+1} = 3 - \dfrac{a_n + 2}{2^{a_n}}$ for all $n \geq 1$. Prove that (a_n) has a finite limit and find it.

629. Let (x_n) be defined by $x_1 = 1$ and $x_{n+1} = \dfrac{x_n^2 + 4x_n + 1}{x_n^2 + x_n + 1}$ for $n \geq 1$. Prove that (x_n) has a finite limit and find it.

630. Let (u_n) be defined by $u_1 = 1$ and $u_{n+1} = \dfrac{2}{u_n} + \dfrac{\sqrt{3}}{u_n^2}$ for $n \geq 1$. Prove that (u_n) does *not* have a finite limit.

631. Let (x_n) satisfy $x_1 = 0,\ x_2 = 1$ and $x_{n+1} = \dfrac{3x_{n-1} + 2}{10x_n + 2x_{n-1} + 2}$ for $n \geq 2$. Prove that (x_n) has a finite limit and find it.

632. Let (x_n) be defined by $x_1 = 1$ and

$$x_{n+1} = \frac{\pi}{8}\left(\cos x_n + \frac{\cos 2x_n}{2} + \frac{\cos 3x_n}{3}\right).$$

Prove that (x_n) has a finite limit.

633. Let (x_n) be a real sequence defined by $x_1 = 3$ and $x_n = \dfrac{n+2}{3n}(x_{n-1} + 2)$ for $n \geq 2$. Prove that the sequence has a finite limit as $n \to +\infty$ and find it.

634. Let $(u_n)_{n\in\mathbb{N}}$ be defined by $u_0 = u_1 = 1$ and

$$u_{n+1} = \frac{1}{2}u_{n-1} + \frac{n+1}{2n+3}\sqrt{u_n + 2}, \quad n = 1, 2, 3, \ldots$$

Prove that $(u_n)_{n\in\mathbb{N}}$ has a finite limit and find $\lim u_n$.

635. Let (u_n) satisfy $u_1, u_2 \in (0, 1)$ and $u_{n+2} = \dfrac{1}{5}u_{n+1}^3 + \dfrac{4}{5}\sqrt[3]{u_n}$ for $n = 1, 2, \ldots$. Prove that (u_n) has a finite limit and find it.

636. Let (x_n) be defined by $x_1 = x_2 = 1$ and $x_{n+2} = \log_3(3x_{n+1} + 75) - \log_2(x_n + 2)$, $n = 1, 2, 3, \ldots$. Prove that (x_n) has a finite limit and find $\lim x_n$.

637. Let a be a real number and let $\{x_n\}$ satisfy $x_1 = 1,\ x_2 = 0$, $x_{n+2} = \dfrac{x_n^2 + x_{n+1}^2}{4} + a$ for all $n \geq 1$. Find the largest real number a such that $\{x_n\}$ converges.

638. Let (x_n) be defined by $x_1 = 2$ and $x_{n+1} + \dfrac{x_n^2}{2} = \dfrac{2n+1}{2n} + x_n$ for $n \geq 1$. Prove that (x_n) has a finite limit and find it.

639. Let (u_n) satisfy $u_1 = \dfrac{15}{8},\ u_2 = 2$, and

$$u_{n+2} + \frac{1}{2} = \sqrt{u_{n+1}^2 + u_n + \frac{n^2}{4n^2 - 1}}, \quad \forall n \in \mathbb{N}.$$

Prove that (u_n) converges and find its limit.

640. Let (x_n) be defined by $x_0, x_1, x_2 \in (0,1)$ and $3x_{n+3} = x_n^2 + x_{n+2}^2$ for $n = 0, 1, 2, \ldots$. Prove that (x_n) has a finite limit and find it.

641. Let a, b be positive reals and let (u_n) be defined by $u_1 = a,\ u_2 = b$, $u_{n+2} = 3\sqrt[5]{u_{n+1}} + 13\sqrt[5]{u_n}$ for $n \in \mathbb{N}^*$. Prove that (u_n) has a finite limit and find it.

642. Let $a, b \in (0,4)$ be real numbers and let (x_n) be defined by $x_0 = a,\ x_1 = b$,

$$x_{n+2} = \frac{2(x_{n+1} + x_n)}{\sqrt{x_{n+1}} + \sqrt{x_n}}, \quad \forall n \in \mathbb{N}.$$

Find the finite limit of (x_n), if it exists.

643. Let (u_n) satisfy $u_0 = 1,\ u_1 = 3$ and

$$u_{n+1}^2 + 4u_{n+1} = u_n + u_{n-1} + 2\sqrt{(u_n + 2)(u_{n-1} + 2)}.$$

Prove that (u_n) has a finite limit and find it.

644. Given four positive real numbers a, b, A, B. Let (x_n) be defined by $x_1 = a,\ x_2 = b$,

$$x_{n+2} = A\sqrt[3]{x_{n+1}^2} + B\sqrt[3]{x_n^2}, \quad \forall n \in \mathbb{N}^*.$$

Prove that (x_n) has a finite limit and find it.

645. Let (x_n) be defined by $x_0, x_1, x_2 \in \mathbb{R}$ and $3^{x_n} + 4^{x_{n+2}} = 5^{x_{n+3}}$ for all $n \in \mathbb{N}$. Prove that the sequence has a finite limit and find it.

646. Let (x_n) be defined by

$$x_1 = \frac{1}{2}, \qquad x_n = \frac{1}{2}\left(\sqrt{x_{n-1}^2 + 4x_{n-1}} + x_{n-1}\right), \ \forall n \geq 2.$$

Prove that (y_n) defined by $y_n = \sum_{i=1}^{n} \frac{1}{x_i^2}$ has a finite limit, and find it.

647. Let (x_n) be defined by $x_1 = 1$ and $x_{n+1} = x_n + x_n^2$ for $n \in \mathbb{N}^*$. Find

$$\lim_{n\to+\infty} \left(\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_n}{x_{n+1}}\right).$$

648. Let $\langle a \rangle$ and $\langle b \rangle$ be two infinite sequences of positive integers. Define the sequence $\langle x \rangle$ by

$$x_n = \frac{a_1^{b_1} \cdots a_n^{b_n}}{\left(\frac{a_1 b_1 + \cdots + a_n b_n}{b_1 + \cdots + b_n}\right)^{b_1 + \cdots + b_n}}, \qquad n \geq 1.$$

(a) Prove that $\lim_{n\to\infty} x_n$ exists.

(b) Find all real numbers c such that there exist sequences $\langle a \rangle$ and $\langle b \rangle$ with $\lim_{n\to\infty} x_n = c$.

649. Let (A_n) be defined by

$$A_n = \frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \cdots + \frac{n}{n^2 + n^2}.$$

Find

$$\lim_{n\to\infty} n\left(n\left(\frac{\pi}{4} - A_n\right) - \frac{1}{4}\right).$$

650. Let (a_n) be defined by $a_1 = a_2 = 1$ and $a_{n+1} = a_n + \frac{a_{n-1}}{n(n+1)}$ for $n \geq 2$. Prove that (a_n) converges.

651. Let (x_n) be a sequence of real numbers with $x_1 \in \left(0, \frac{1}{2}\right)$ and $x_{n+1} = 3x_n^2 - 2x_n^3$ for all $n \geq 1$.

(1) Prove that $\lim_{n\to\infty} x_n = 0$.

(2) For $n \geq 1$, let $y_n = x_1 + 2x_2 + \cdots + nx_n$. Prove that (y_n) converges.

652. Let
$$E = \left\{ f \in C^1([0,1], \mathbb{R}) : \; f(0) = 0, \; f(1) = 1 \right\}.$$
Determine
$$\inf_{f\in E} \int_0^1 \left| f'(x) - f(x) \right| \, dx.$$

653. Let (a_n) be the sequence with $a_1 = \dfrac{1}{2}$ and, for each $n \geq 1$, a_{n+1} the unique number in $[0,1]$ satisfying
$$a_n = 3a_{n+1} - a_{n+1}^3.$$

(1) Prove that (a_n) is well-defined and converges.

(2) Let $b_n = (1+2a_1)\left(1 + 2^2 a_2\right)\cdots(1+2^n a_n)$ for $n \geq 1$. Prove that (b_n) converges.

654. Let (x_n) be defined by $x_1 = 1$ and $x_{n+1} = \sqrt{\dfrac{n+2}{n} + \ln x_n}$. Find $\lim_{n\to\infty} x_n$.

655. Let (y_n) be defined by
$$y_{n+1} = \frac{n+1}{2^{n+1}} \left(\frac{2}{1} + \frac{2^2}{2} + \cdots + \frac{2^n}{n} \right).$$
Find $\lim_{n\to\infty} y_n$.

656. Let $a, b \in (0,1)$ and let (u_n) be defined by $u_0 = a$, $u_1 = b$, and $u_{n+2} = \dfrac{1}{3}u_{n+1}^4 + \dfrac{2}{3}\sqrt[4]{u_n}$. Prove that (u_n) converges and find its limit.

657. Let $f : [0,1] \to \mathbb{R}$ be an integrable function such that
$$\int_0^1 f(x)\, dx = 1$$
and
$$\int_0^1 x^2 f(x)\, dx = 1.$$
Prove that
$$\int_0^1 f^2(x)\, dx \geq 6.$$

658. Let $f : [0,1] \to \mathbb{R}$ be an integrable function. Now set

$$\mu_k := \int_0^1 x^k f(x)\, dx, \qquad k = 0, 1, 2.$$

Prove that

$$\frac{1}{3}\int_0^1 f^2(x)\, dx \geq 10\mu_2(\mu_0 + \mu_1) - \mu_1(8\mu_0 + 9\mu_1).$$

659. If $f : (0,\infty) \to \mathbb{R}$ is a convex function, show that

$$\lim_{n\to\infty} \int_{2n\pi}^{2(n+1)\pi} f(x)\sin(x)\, dx = 2\pi \cdot \sup_{n\in\mathbb{N}} \frac{f(2n) - f(n)}{n}.$$

660. Let $f : [0,1] \to \mathbb{R}$ be a continuous and nonconstant function. We consider the function

$$g : \mathrm{Im}(f) \to \mathbb{R}$$

defined by

$$g(y) = \inf\{x \in [0,1] \mid f(x) = y\}, \qquad \text{for any } y \in \mathrm{Im}(f),$$

where

$$\mathrm{Im}(f) = \{f(x) \mid x \in [0,1]\}.$$

Prove that the function g is continuous if and only if there exists $a \in (0,1)$ such that f is strictly monotonic on the interval $[0,a]$ and

$$\mathrm{Im}(f) = f([0,a]).$$

661. For a nonnegative integer n and a strictly increasing sequence of real numbers

$$t_0, t_1, \ldots, t_n,$$

let $f(t)$ be the corresponding real-valued function defined for $t \geq t_0$ by the following properties:

(a) $f(t)$ is continuous for $t \geq t_0$, and is twice differentiable for all $t > t_0$ other than $t_1, \ldots, t_n$;

(b)

$$f(t_0) = \frac{1}{2};$$

(c)

$$\lim_{t\to t_k^+} f'(t) = 0, \qquad 0 \leq k \leq n;$$

(d) For $0 \le k \le n-1$, we have

$$f''(t) = k+1 \qquad \text{when } t_k < t < t_{k+1},$$

and

$$f''(t) = n+1 \qquad \text{when } t > t_n.$$

Considering all choices of n and $t_0, t_1, \ldots, t_n$ such that

$$t_k \ge t_{k-1} + 1 \qquad \text{for } 1 \le k \le n,$$

what is the least possible value of T for which

$$f(t_0 + T) = 2023?$$

662. We consider a function

$$f : \mathbb{R} \to \mathbb{R}$$

for which there exist a differentiable function

$$g : \mathbb{R} \to \mathbb{R}$$

and a sequence $(a_n)_{n \ge 1}$ of positive numbers, convergent to 0, such that

$$g'(x) = \lim_{n \to \infty} \frac{f(x + a_n) - f(x)}{a_n}, \qquad \forall x \in \mathbb{R}.$$

(a) Give an example of such a function f that is not differentiable at any point $x \in \mathbb{R}$.

(b) Show that if f is continuous on $\mathbb{R}$, then f is differentiable on $\mathbb{R}$.

663. Let $0 < a < b$ be real numbers, and let

$$f, g : [a, b] \to (0, \infty)$$

be continuous functions. Prove that there exists $c \in (a, b)$ such that

$$\left(\frac{1}{f(c)} + \frac{1}{\int_c^b g(t)\, dt} \right) \left(g(c) + \int_a^c f(t)\, dt \right) \ge 4.$$

664. Suppose

$$f : [a, b] \to [1, \infty)$$

is a continuous function with $0 < a \le b$. Then

$$n(b-a)^{n-1} \int_a^b f(x)\, dx \le (n-1)(b-a)^n + \left(\int_a^b f(x)\, dx \right)^n.$$

665. Show that if $f : [0,1] \to \mathbb{R}$ is continuous and

$$\int_0^1 x(1-x)^2 f(x)\,dx = 0,$$

then there exists $\mu \in (0,1)$ such that

$$\int_0^\mu x^2 f(x)\,dx = \frac{\mu}{2} \int_0^\mu x f(x)\,dx.$$

Show that for any $\epsilon > 0$, there are f, μ as in (a) such that f is not identically zero and $\mu < \epsilon$.

666. Let $(a_n)_{n\geq 1} \subset (0,\infty)$ satisfy

$$a_n \longrightarrow 0 \qquad (n \to \infty),$$

and assume that there exists a constant $c > 0$ such that

$$|a_{n+1} - a_n| \leq c\, a_n^2 \qquad \text{for all } n \geq 1.$$

Prove that there exists $d > 0$, depending only on the sequence, such that

$$a_n \geq \frac{d}{n} \qquad \text{for every } n \geq 1.$$